A (forgotten) analytical infrastructure to neoclassical economics: *La théorie générale des surplus* by Maurice Allais

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Abstract

Maurice Allais' works on the theory of surplus have received very little attention. The present paper argues that this has deprived (and still deprives) microeconomic analysis from: (1) the basic observation that, within a static perspective, Pareto-efficiency does not result from competition but simply from the exhaustion of gains drawn from informed voluntary transactions; (2) a useful conceptual framework well equipped to provide a consistent analytical infrastructure to neoclassical economics. To make these points clear, Allais' theory is reformulated with modern concepts and relieved from useless complications. Doing so, the present paper helps the assessment of Allais' theory as extending the marshallian analysis within a general interdependence framework. Within a static analysis, competition influences distribution, not efficiency.

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1 Introduction

This paper reformulates Maurice Allais' *Théorie générale des surplus*[1] (TGS hereafter) in a standardized way so as to (re-)assert its relevance and unifying potential for microeconomic theory. The intention is to restore a useful conceptual and analytical equipment which passed almost unnoticed[2] at its time (early 80's) although it fills gaps that still exist today. The central idea is that microeconomic theory could profitably be reorganized around a concept of surplus which takes into account the principle of general interdependence, rather than by reference to the concept of walrasian general equilibrium (WGE). The reason is that, to the extent that the concept of surplus goes with *out-of-equilibrium* analyses, it is far more inclusive and open to modern developments (strategic behaviors) than standard walrasian theory. More precisely, rather than considering the economic system at the scale of the market (or the individual), the TGS considers it at the scale of the *transaction*. Not surprizingly, insofar as it allows a position of exteriority with respect to markets, it is particularly well suited to thinking about economic institutions with a more open mind.

The aim of the present paper is first to provide a clarified exposition of the TGS using standard notions. Despite its achievement, a few ambiguities, and unnecessary complications remain in Allais' original text; addressing this issue should facilitate the assessment of the TGS, and to highlight its scope. This paper further adds some results and proofs neglected by Allais, and put key results into perspective. Some developments posterior to the TGS are also included which clarify or extend Allais' analysis.

1.1 Welfare analysis

Surplus is a measure of the gains resulting from informed voluntary economic activity. the TGS provides an operational definition of it with no recourse to: any cardinal notion of utility; any given price system; any generalized assumptions of continuity, derivability, nor convexity. Allais' surplus[3] allows to analyze within a general interdependence framework: out-of-equilibrium microeconomic behaviors, the economic processes induced by voluntary exchange and cooperation, and the conditions for *Pareto-efficiency*.

The basic framework is that of an Arrow-Debreu economy: a given list of private goods; a given set of individuals; given endowments, preferences, and technologies. From any initial allocation of resources, Allais' concept of surplus, as measured in a good of reference, is the quantity of this good that

[1] In what follows, one refers to the 1989 (second) publishing of the original text: Allais, M. (1989), *La Théorie Générale des Surplus*, Presses Universitaires de Grenoble.

[2] Beyond Guesnerie (1984)'s report on the TGS in the *Journal of Political Economy* and some tributes connected to the Nobel Price, a quick Jstor search leads to only one paper mentioning the TGS as an input for original research (Diewert, 1981). Same exercice conducted with *Googles Scholar* displays a series of articles of History of economic thought. Enlarging the search, one finds some papers in the *Journal of Mathematical Economics* (Luenberger, 1996) and *Economic Theory* (Courtault and Tallon, 2000).

[3] He terms his concept *distributable surplus* ("surplus distribuable").

can be *released* (made available) by a reallocation leaving each individual utility level constant. Allais shows that an allocation is Pareto-efficient if and only if, whatever the good of reference in which it is measured, surplus is negative or nul for any feasible reallocation. The *loss* associated to a given allocation is the *maximal surplus* releasable through a feasible allocation.

Now, assume there exists a perfectly divisible good desired by all individuals in the economy (whatever their endowment in the good under consideration or any other good) and let's call "money" this particular good. Surplus as measured in "money" is negative or nul if and only if it is negative or nul as measured in any other good of reference. If follows that an allocation is Pareto-efficient if and only if the surplus as measured in "money" is negative or nul for any feasible reallocation. Allais' "monetary" concept of loss provides a consistent index of inefficiency.

Contrary to most alternative concepts, Allais' surplus does not lies on any given price system; moreover, it adresses the problems encountered by the literature on surplus, which confined to partial welfare analyses.[4] Allais' concept is nevertheless close to what Hicks (1956) refers to as "compensating surplus."

1.2 Positive analysis

Beyond welfare analysis, the aim of the TGS is to break with a positive interpretation of the walrasian model, as well as to provide an alternative theory incorporating the neoclassical legacy. Since no given system of prices is available to individuals, they cannot be considered as maximizing utility subject to budget constraint. As a positive analysis, the TGS is based on the assumption that individuals are *surplus-seekers* (for their own account). It follows that, provided a transaction is informed and voluntary, it is necessarily Pareto-improving. Hence, a decentralized process of voluntary exchanges and cooperation can be expected to drive the economy to a "least-loss" allocation, if not to a Pareto-efficient allocation (in the absence of any transactional obstacles). Apart from the respect of individual willingness to trade and the right to property, the analysis does not rely on any specific institutional setting: exchanges may be bilateral or multilateral; there might be integrated organizations or not; in case of market transactions, individuals may be price takers or not, etc. In particular, and implicit in the absence of any given price system, there is no assumption about to the degree of competition within the economy. This structural parcimony makes the TGS a valuable infrastructure to microeconomic analysis. It provides a proper assessement of the problem economic agents (embedded individuals as well as economists) have to solve, namely to identify and make possible P-improving transactions or, as Allais puts it: to search, to realize, and to distribute surplus.

[4] See Currie et al. (1971).

1.3 Out-of-equilibrium analysis and "money"

What the TGS achieves has lots to do with the consideration of a particular good, simply refered to as "money." Since the TGS is not intended to provide a theory of money,[5] a precautionary choice is made hereafter to always keep quotations marks. And yet, it is reasonable to think of the TGS as considering a *fiat money*, *i.e.* an object logically inconsistent with walrasian theory of general equilibrium. The usual argument is that, insofar as fiat money has no intrinsic utility, there is no point for individuals to retain cash balances: at *equilibrium*, the value of money is zero.[6] The difficulty to make room to money within the walrasian framework is thus closely linked to the emphasis put on equilibrium situations. Two primary features of the TGS make a direct integration of "money" in (all) utility functions an admissible short-cut: first, it applies out-of-equilibrium; second, it does not rely on any assumption of perfect information. Allais' point is the same as that of Marshall in *The Principles*, that is to analyse the functioning of an economy which happens to be monetary. In this respect, the utility of "money" (something desired by everyone in every circumstances) is its near-universal acceptability in exchange for other commodities,[7] which is a concern for any individual who *does not know* whether the economy is at equilibrium. The way "money" is treated in the TGS is thus internally consistent, provided perfect information is not assumed.

1.4 The contribution of the present paper

The main purpose of the present paper is to promote understanding and assessment of the TGS. This sometimes requires to take some distance from the original text or to develop some meaningful implications.

Section 2 is devoted to the exposition of the TGS with weak assumptions. As compared to Allais' treatment, the present paper adds some formal definitions and some proofs for results improving on the literature on surplus - for instance, as regards the fact that Allais' surplus indeed lies on ordinal preferences.[8] It also provides an extensive discussion of the relation between loss-reducing and Pareto-improving reallocations. It is shown that, as an index of inefficiency, loss defines a preorder on allocations which is "less incomplete" than Pareto-improvement: any P-improving reallocation reduces loss but a reallocation reducing loss is not necessarily P-improving.

In Section 3, the "monetary" formulation of the TGS is presented. In this section, an important precaution, overlooked by Allais, is taken as regards the integration of "money" in transformation technologies. The choice is made to

[5] But rather a monetary theory of surpluses.

[6] See Ostroy and Starr (1990) for further developments.

[7] Most of the discussion by Shapley and Shubik (1977) of the circumstances under which one can argue for a direct integration of money within utility functions could be repeated here.

[8] Filling the gap between Marshall's surplus (ordinary demand curve but cardinal concept of utility) and Hicks equivalent income (ordinal concept of utility but compensated demand curve).

leave "money" apart from any transformation operation. The point is twofold: first, to discard any possibility to produce some "money" (confusion between producing "money" and making "monetary" profit); second, to avoid the mistake consisting in treating transformation technologies as individuals, the well-being of whom deserve to be considered when assessing collective welfare. Apart from this clarification, little is added to original text except the highlighting that changes in the distribution of "money" among individuals, all other things being equal, cannot release surplus at the collective level; a corollary is that it does not destroy surplus neither.

Section 4 operates the sliding from a normative interpretation of the TGS to a positive one: general equilibrium is understood as resulting from the exhaustion of a surplus releasing process.

Section 5 connects the TGS with the marginalist analysis considering infinitesimal reallocations and assuming differentiability. One task accomplished in this section is to clarify Allais' analysis through a direct reasoning in terms of (marginal) subjective valuations *i.e.* marginal rates of substitution of goods for "money". This simplifies interpretation to the extent that the surplus associated to an infinitesimal reallocation directly results from individuals' valuations; besides, this rehabilitates Marshall's concept of surplus in the case of infinitesimal transactions. It also highlights the symmetry between the concept of marginal valuation, on the side of individuals, and that of marginal transformation loss, on the side of transformation technologies, which most notably translates into a "decreasing marginal return in surplus" as a second order condition for an allocation to be P-efficient.

This note can be read independantly from the original text of Allais. However, to facilitate comparisons, references to the original text are provided.

2 The TGS

With proper information, when there are no external costs[9], voluntary exchange and cooperation brings the economy in a "preferable state." This is a qualitative statement: the concept of surplus aims to provide a tool to assess this statement on a quantitative basis.

2.1 The framework

Individuals[10] are indexed by $i \in \mathcal{I} = \{1, ..., I\}$; transformation technologies are indexed by $j \in \mathcal{J} = \{1, ..., J\}$; goods are indexed by $n \in \mathcal{N} = \{1, ..., N\}$. There are N *private goods* (or services) in the economy, among them one good of reference $\bar{n}$. A vector of quantities is denoted x and $\mathbf{x'} = (x_1, ..., x_N)$.

[9] Associated to the consumption of private goods.

[10] Consumers and resources holders, including any decision unit whoses welfare is considered *per se* in the analysis.

2.1.1 Individual preferences

The utility concept used is a purely ordinal one. Individual i's preferences over $\mathbf{x}_i \in \mathbb{R}^N$ are represented by the utility function $u_i(.)$ defined by $u_i = u_i(\mathbf{x}_i)$ where: u_i is a subjective measure of well-being; $x_{in} > 0$ represents a (final) consumption (*i.e.* a quantity drawn from "the economy"), and $x_{in} < 0$ an individual production (or service delivered to "the economy"), of good n by individual i.[11] For all $n \in \mathcal{N}$, $u_i(\mathbf{x}_i)$ is assumed strictly increasing.[12]

2.1.2 Transformation sets

Each transformation technology j is characterized by the function $f_j(.)$ defined over the set of *transformation plans* $\mathbf{x}_j \in \mathbb{R}^N$. The following conventions apply: $x_{jn} > 0$ represents a quantity of input n used (*i.e.* drawn from "the economy") in j; $x_{jn} < 0$ represents a quantity of output produced (*i.e.* injected into "the economy") from j. Note that this convention is the opposite to the usual one, but similar to what is done for individual.

A transformation plan $\mathbf{x}_j$ is feasible $\Leftrightarrow f_j(\mathbf{x}_j) \geq 0$. $f_j(\mathbf{x}_j) > 0$ means that the plan $\mathbf{x}_j$ involves wastage in transformation (resources dissipation which benefit to no individual) or that some outputs are diverted from "the economy" (to some individuals outside "the economy" *i.e.* whose welfare is not considered in the analysis). $l_j = f_j(\mathbf{x}_j)$ is an (unobserved) index of *transformation loss* attached to the operating of technology j. Under this convention, $f_j(\mathbf{x}_j) = 0$ means that the transformation operation occurs with no loss, *i.e.* without resources dissipation. Figure 1 illustrates previous concepts.

In Figure 1, the good n is an output, the good $\underline{n}$, an input; the technology j allows to produce a quantity $-x_{jn} > 0$ of good n. The condition $f_j\left(x_{jn}, x_{j\underline{n}}; \mathbf{x}_j^{\neg n,\underline{n}}\right) = l_j$ involves that, assuming quantities are continuous and $f_j(.)$ is derivable, the marginal productivity of input $\underline{n}$ (as measured in good n) is given by $\frac{f'_{j\underline{n}}(\mathbf{x}_j)}{f'_{jn}(\mathbf{x}_j)} > 0$.[13]

2.1.3 Feasible allocations and reallocations

An *allocation* is a list $a = \left((\mathbf{x}_i)_{i\in\mathcal{I}}, (\mathbf{x}_j)_{j\in\mathcal{J}}\right)$. It is *feasible* if and only if

$$\sum_{i\in\mathcal{I}} \mathbf{x}_i + \sum_{j\in\mathcal{J}} \mathbf{x}_j \leq \overline{\mathbf{x}},$$
$$f_j(\mathbf{x}_j) \geq 0 \text{ for all } j \in \mathcal{J},$$

[11] Note that, with this convention, leasure and labor should be seen as two distinct goods. Previous convention requires: $x_{\text{leasure}} > 0$ and $x_{\text{labor}} < 0$ with $x_{\text{leasure}} - x_{\text{labor}} \leq$ time endowment.

[12] This assumption is useful most notably to the extent that no exogenous restriction is made on the set of possible plans (no lower bound condition is set). It is dropped once "money" is introduced.

[13] Or, more generally, $\frac{f'_{j\underline{n}}(\mathbf{x}_j)}{f'_{jn}(\mathbf{x}_j)} - \frac{dl_j}{f'_{jn}(\mathbf{x}_j)}$ if $dl_j \neq 0$.

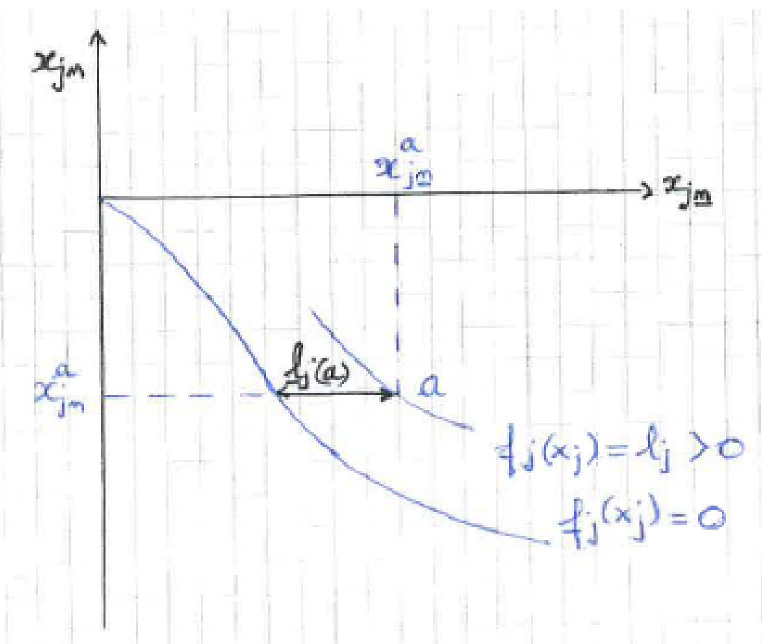

Figure 1: Transformation function and transformation loss

where $\overline{\mathbf{x}}$ gives the economy's initial global resources of each good. Let $\mathcal{A}$ denotes the set of feasible allocations. Given an allocation $a \in \mathcal{A}$, a *reallocation* is a list of variations $\Delta a = \left((\Delta \mathbf{x}_i)_{i \in \mathcal{I}}, (\Delta \mathbf{x}_j)_{j \in \mathcal{J}} \right)$ from the allocation a. Starting from an allocation such that $\sum_{i \in \mathcal{I}} \mathbf{x}_i + \sum_{j \in \mathcal{J}} \mathbf{x}_j = \overline{\mathbf{x}}$, and $f_j(\mathbf{x}_j) = 0$ for all $j \in \mathcal{J}$, a reallocation Δa is *feasible* if and only if

$$\sum_{i \in \mathcal{I}} \Delta \mathbf{x}_i + \sum_{j \in \mathcal{J}} \Delta \mathbf{x}_j \leq \mathbf{0},$$

$$f_j(\mathbf{x}_j + \Delta \mathbf{x}_j) \geq 0 \text{ for all } j \in \mathcal{J}.$$

Previous relations mean that, for any given good, any increase of consumption by an individual or of intermediate consumption in a transformation operation requires: a reduction in the consumption of some other individual, and/or an increase in production.

2.2 Basic concepts

The main concepts of the analysis are now stated under the weakest assumptions: quantities may be continuous or not; transformation sets and preferences convex or not.

2.2.1 Surplus

Allais gives the following informal definition of surplus. From an initial allocation, for any subset of individuals with given endowments, the *surplus* corresponding to a given reallocation Δa, as measured in any good of reference $\bar{n}$, is the quantity $\Delta s_{\bar{n}}$ of this good that can be *released* (made available) from Δa under the threefold condition that: the quantity used of each resource is at most equal to its initial level; the services delivered by the subset of individuals under consideration to the rest of the economy is at least equal to its initial level; *each individual in the subset under consideration gets a utility at least equal to its initial level.*[14,15] A formal definition of surplus can be adapted in the case where $u_i(\mathbf{x}_i)$ is strictly increasing for all $n \in \mathcal{N}$ from that of the *benefit function* as introduced in Luenberger (1992).

Individual surplus

Definition 1 *For any individual $i \in \mathcal{I}$ with initial plan $\mathbf{x}_i \in \mathbb{R}^N$, the individual surplus $\Delta s_{i\bar{n}}$, as measured in any good of reference $\bar{n}$, associated to the change $\Delta \mathbf{x}_i = (\Delta \mathbf{x}_i^{\neg \bar{n}}, \Delta x_{i\bar{n}})$ is*

$$\Delta s_{i\bar{n}} \equiv \max \left\{ \Delta \sigma_{i\bar{n}} \in \mathbb{R} \mid u_i \left(\mathbf{x}_i^{\neg \bar{n}} + \Delta \mathbf{x}_i^{\neg \bar{n}}, x_{i\bar{n}} + \Delta x_{i\bar{n}} - \Delta \sigma_{i\bar{n}} \right) \geq u_i(\mathbf{x}_i) \right\}.$$

[14] Allais, TGS, §113.

[15] One can think of $\Delta s(a)$ as the variation of an unobservable index $s(a)$ measuring the welfare level (expressed as a quantity of some private good) associated to a by comparison to an hypothetical state in which each consumer leaves in complete autarky: initial global resources are entirely distributed among them, no exchange nor production occurs.

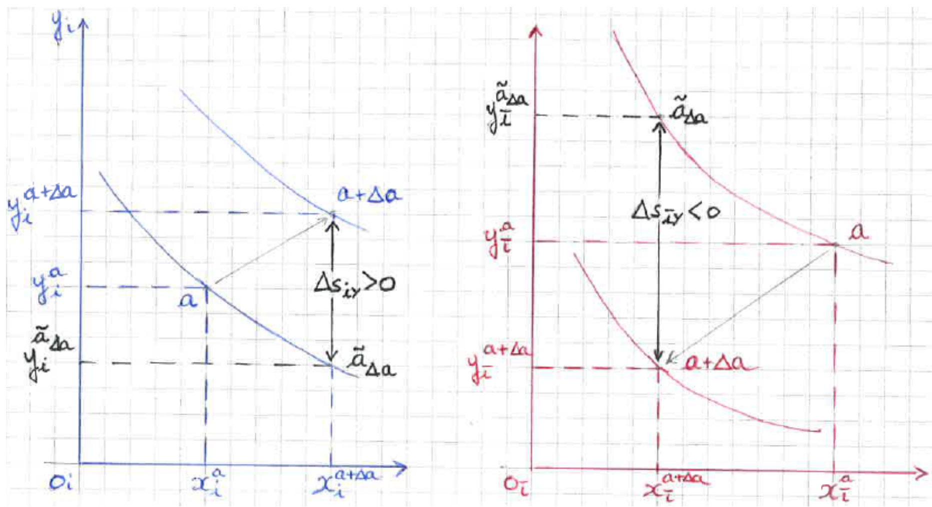

Figure 2: Individual surplus corresponding to a change.

Note that, since $u_i(\mathbf{x}_i)$ is strictly increasing with respect to $x_{i\bar{n}}$, $\Delta s_{i\bar{n}}$ always exists.[16] The interpretation is straightforward:

- If $u_i\left(\mathbf{x}_i^{\neg\bar{n}} + \Delta\mathbf{x}_i^{\neg\bar{n}}, x_{i\bar{n}} + \Delta x_{i\bar{n}}\right) > u_i(\mathbf{x}_i)$, $\Delta s_{i\bar{n}} > 0$ is the highest amount of the good of reference agent i would be willing to give up in exchange for implementing the change $\Delta\mathbf{x}_i$;

- If $u_i\left(\mathbf{x}_i^{\neg\bar{n}} + \Delta\mathbf{x}_i^{\neg\bar{n}}, x_{i\bar{n}} + \Delta x_{i\bar{n}}\right) < u_i(\mathbf{x}_i)$, $\Delta s_{i\bar{n}} < 0$ is the smallest amount of the good of reference agent i would call for so as to accept the change $\Delta\mathbf{x}_i$.

Note the difference in nature between variations $\Delta\mathbf{x}_i$ and the quantity $\Delta s_{i\bar{n}}$: $\Delta\mathbf{x}_i$ is a change that is *actually* under consideration whereas $\Delta s_{i\bar{n}}$ is a *virtual* quantity measuring the attitude of agent i as regards the implementation of $\Delta\mathbf{x}_i$. If $\Delta s_{i\bar{n}} > 0$ ($\Delta s_{i\bar{n}} < 0$), the change $\Delta\mathbf{x}_i$ is said to *distribute* a positive (resp. negative) individual surplus; if $\Delta s_{i\bar{n}} = 0$, the change $\Delta\mathbf{x}_i$ distributes no surplus.

Figure 2 illustrates the measure of individual surplus associated to a change; both the case of a positive surplus (lefthand), and that of a negative one (righthand) are considered. Figure 2 considers two consumption goods (the quantities of which are denoted x and y). Note that, although two indifference curves are plotted, only the one passing through the initial allocation is required.

Collective surplus

[16] Allais fails to raise the issue of existence. Luenberger deals with it with a definition of surplus which account for the case of non-existence of some $\Delta\sigma_{in}$.

Definition 2 *Given an initial allocation $a \in \mathcal{A}$, for any subset $\mathcal{I}_{\circledS} \subseteq \mathcal{I}$ of individuals, the surplus corresponding to a given reallocation Δa is*

$$\Delta s_{\circledS \bar{n}} \equiv \sum_{i \in \mathcal{I}_{\circledS}} \Delta s_{i\bar{n}} - \sum_{i \in \mathcal{I}_{\circledS}} \Delta x_{i\bar{n}}$$

such that:
(i) Δa is feasible;
(ii) $\forall i \in \mathcal{I}, i \notin \mathcal{I}_{\circledS}, \Delta \mathbf{x}_i \geq \mathbf{0}$;
(iii) $\forall i \in \mathcal{I}_{\circledS}, \Delta s_{i\bar{n}} \equiv \max \{\Delta \sigma_{i\bar{n}} \in \mathbb{R} \mid u_i (\mathbf{x}_i^{\neg \bar{n}} + \Delta \mathbf{x}_i^{\neg \bar{n}}, x_{i\bar{n}} + \Delta x_{i\bar{n}} - \Delta \sigma_{i\bar{n}}) \geq u_i (\mathbf{x}_i)\}$.

Three cases can be distringuished at the scale of $\mathcal{I}_{\circledS}$:

- If $\sum_{i \in \mathcal{I}_{\circledS}} \Delta x_{i\bar{n}} = 0$, the surplus is fully distributed $\Delta s_{\circledS \bar{n}} = \sum_{i \in \mathcal{I}_{\circledS}} \Delta s_{i\bar{n}}$;
- If $\sum_{i \in \mathcal{I}_{\circledS}} \Delta x_{i\bar{n}} < 0$ and $\sum_{i \in \mathcal{I}_{\circledS}} \Delta s_{i\bar{n}} \neq 0$, the surplus is partially distributed;
- If $\Delta s_{i\bar{n}} = 0$ for all $i \in \mathcal{I}_{\circledS}$, some surplus may be released but it remains fully retained, $\Delta s_{\circledS} = -\sum_{i \in \mathcal{I}_{\circledS}} \Delta x_{i\bar{n}}$.

An important feature of Allais' analysis is that it can be applied at any scale from the bilateral transaction to a reallocation impacting all individuals in the economy. The total surplus associated to a reallocation is the sum of "local" surplus ("local" surplus are additive) provided that the second condition in definition 2 is met.

Example 3 *Figure 3 illustrates the concept of surplus (as measured in good* Y*) for a subset of two individuals. The point is to consider reallocations which do not reduce individuals' welfare while releasing some positive surplus. Consider the two lefthand side graphs first, which illustrate a reallocation releasing a positive surplus without distributing it. The initial allocation a is such that $x_1^a + x_2^a = \bar{x}\,(= 16)$ and $y_1^a + y_2^a = \bar{y}\,(= 16)$, while the final allocation $\tilde{a}$ is such that:*

$$\begin{aligned} \tilde{x}_1^a + \tilde{x}_2^a &= \bar{x}\,(= 16)\,, \\ \tilde{y}_1^a + \tilde{y}_2^a\,(= 14) &< \bar{y}, \end{aligned}$$

and yet, $u_1|_a = u_1|_{\tilde{a}}$ and $u_2|_a = u_2|_{\tilde{a}}$. The released surplus is

$$\Delta s_{\mathrm{Y}} = -\Delta y = \bar{y} - (\tilde{y}_1^a + \tilde{y}_2^a) = 2.$$

The Edgeworth diagram on the righthand side illustrates the case in which surplus is distributed. In this case:

$$\begin{aligned} y_1^{a+\Delta a} + y_2^{a+\Delta a} &= y_1^a + y_2^a = \bar{y}, \\ x_1^{a+\Delta a} + x_2^{a+\Delta a} &= x_1^a + x_2^a = \bar{x}, \end{aligned}$$

but still

$$\Delta s_{\mathrm{Y}} = \Delta s_{1\mathrm{Y}} + \Delta s_{2\mathrm{Y}} = 1 + 1 = 2.$$

It is noteworthy that what is done of the surplus released does not impact its amount. Nevertheless, releasing surplus without distributing it, means that corresponding reallocation is "size reducing".

Claim 4 $\Delta s_{Ⓢ\bar{n}}$ *is invariant with respect to monotonous increasing transformations of utility functions.*

Proof. See the appendix. ■

Allais' surplus is an ordinal concept. It thus requires no assumption of transferable utility (which involves absent income effect).

2.2.2 Loss

The (deadweight) *loss* associated to some allocation, as measured in some good of reference, is the maximal quantity of that good which can be released through a reallocation *i.e.* the maximal releasable surplus.

Definition 5 *Given* $a \in \mathcal{A}$, *for any subset of individuals* $\mathcal{I}_{Ⓢ} \subseteq \mathcal{I}$, *the loss at the scale of* $\mathcal{I}_{Ⓢ}$ *as measured in some good of reference* $\bar{n}$, *is defined by*

$$l_{Ⓢ\bar{n}}(a) \equiv \Delta^* s_{Ⓢ\bar{n}}(a) = \max_{\Delta a \text{ } feasible} \Delta s_{Ⓢ\bar{n}}(a).$$

Figure 4 illustrates the concept of loss in the case of a 2 individuals × 2 goods pure exchange economy. Loss, as measured in good Y, is the maximum vertical distance between indifference curves: tangent lines must be parallel.

Since *statu quo* is always an option, for any $a \in \mathcal{A}$ and $\bar{n} \in \mathcal{N} : l_{\bar{n}}(a) \geq 0$. Since resources are finite and utility function are strictly increasing, for any $a \in \mathcal{A}$ and $\bar{n} \in \mathcal{N} : l_{\bar{n}}(a)$ is finite. It is a function of initial utility levels $(u_i(\mathbf{x}_i))_{i \in \mathcal{I}_{Ⓢ}}$ and of global initial resources $\bar{\mathbf{x}}_{Ⓢ}$ among the subset of indivduals under consideration.

Let $l_{\bar{n}}(a)$ denote the loss, at the scale of the economy, associated to $a \in \mathcal{A}$, as measured in some good of reference $\bar{n} \in \mathcal{N}$.

Corollary 6 $l_{\bar{n}}(a)$ *is invariant with respect to any increasing monotonous transformations of utility functions.*

Proof. A corollary of claim 4. ■

Below, unless explicitly mentioned, the analysis is considered at the scale of the whole economy (subscript Ⓢ is removed). Nevertheless, it can always be restricted to some subset of individuals/technologies by introducing a condition such as (ii) of definition 2.

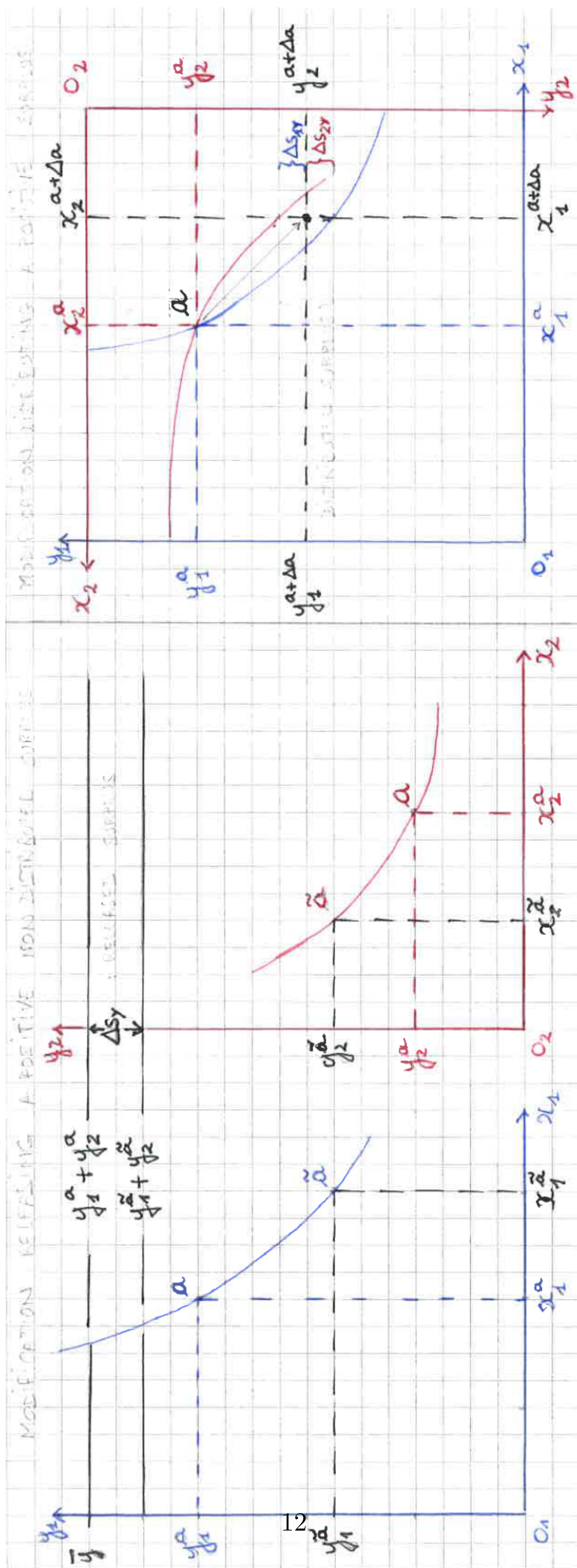

Figure 3: Non-distributed or distributed surplus in an exchange economy.

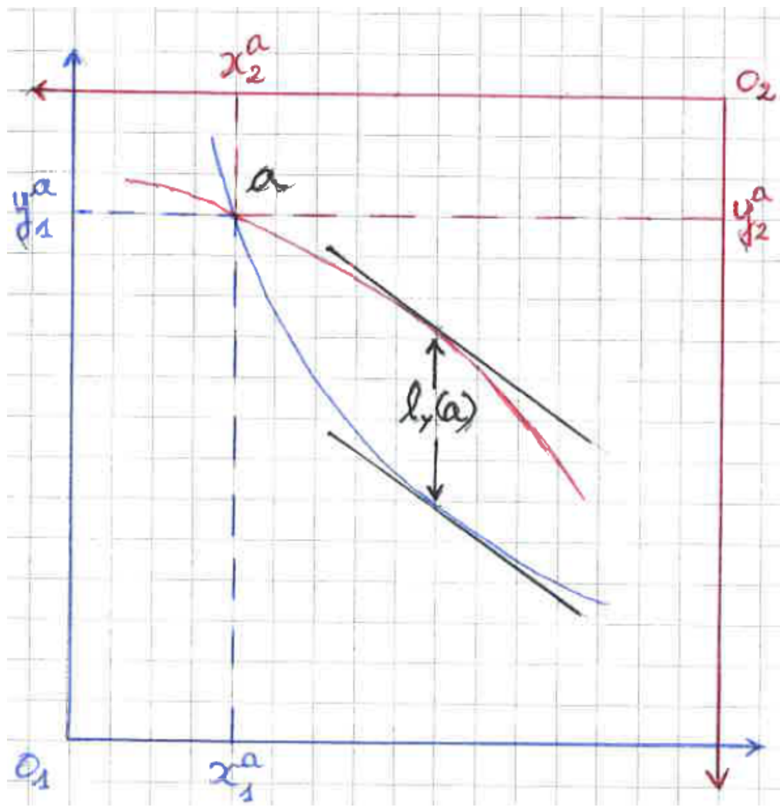

Figure 4: The (deadweight) loss in an exchange economy.

2.3 Pareto-efficiency

Definition 7 *$a^* = \left((\mathbf{x}_i^*)_{i\in\mathcal{I}}, \left(\mathbf{x}_j^*\right)_{j\in\mathcal{J}} \right)$ is Pareto-efficient $\Leftrightarrow$ for all $i \in \mathcal{I}$:*

$$\mathbf{x}_i^* \in \arg\max_{\mathbf{x}_i\in\mathbb{R}^N} \left\{ u_i(\mathbf{x}_i) \mid u_{\bar{\imath}}(\mathbf{x}_{\bar{\imath}}) = u_{\bar{\imath}}(\mathbf{x}_{\bar{\imath}}^*) \text{ for all } \bar{\imath} \in \mathcal{I} - \{i\} \right\}.$$

Note that $a^* \in \mathcal{A}$ Pareto-efficient at the scale of the economy $\Rightarrow \sum_{i\in\mathcal{I}} \mathbf{x}_i^* + \sum_{j\in\mathcal{J}} \mathbf{x}_j^* = \mathbf{0}$, and $f_j\left(\mathbf{x}_j^*\right) = 0$ for all $j \in \mathcal{J}$.

Proposition 8 *$a^* \in \mathcal{A}$ Pareto-efficient at the scale of the economy $\Leftrightarrow l_n(a^*) = 0$ for any good of reference $n \in \mathcal{N}$.*

Proof. See in appendix. ■

Corollary 9 *$a^* \in \mathcal{A}$ Pareto-efficient at the scale of the economy $\Rightarrow a^*$ Pareto-efficient for any subset of individuals.*

Corollary 10 *$a \in \mathcal{A}$ Pareto-efficient at the scale of a subset $\mathcal{I}_{Ⓢ} \subseteq \mathcal{I} \Leftrightarrow l_{Ⓢn}(a) = 0$ for any $n \in \mathcal{N}$.*

Corollary 11 *$a \in \mathcal{A}$ Pareto-inefficient $\Rightarrow$ there exists $n \in \mathcal{N}$ such that $l_n(a) > 0$.*

See figure 5 for an illustration in the case of a 2 individuals $\times$ 2 goods pure exchange economy.

If $a \in \mathcal{A}$ is P-inefficient, let $\Delta^* a$ denote a reallocation such that $\Delta^* a \in \arg\max_{\Delta a \text{ feasible}} \Delta s_n(a)$ for some $n \in \mathcal{N}$. In general, $a + \Delta^* a$ is not P-efficient. See figure 5 for an example in the case of a 2 individuals $\times$ 2 goods pure exchange economy.

Let $\mathcal{R}(a)$ denote the set of *individually rational* reallocations as starting from $a \in \mathcal{A}$ that is

$$\mathcal{R}(a) = \{\Delta a \text{ feasible} \mid \forall i \in \mathcal{I}, u_i(\mathbf{x}_i + \Delta\mathbf{x}_i) \geq u_i(\mathbf{x}_i)\}.$$

From $a \in \mathcal{A}$, a reallocation Δa is Pareto-improving if and only if $\Delta a \in \mathcal{R}(a)$ and $\exists i \in \mathcal{I}$ such that $u_i(\mathbf{x}_i + \Delta\mathbf{x}_i) > u_i(\mathbf{x}_i)$.

Claim 12 *From $a \in \mathcal{A}$, Δa is Pareto-improving $\Rightarrow l_n(a + \Delta a) < l_n(a)$ for any $n \in \mathcal{N}$.*

Proof. See the appendix. ■

The converse is false: $l_n(a + \Delta a) < l_n(a)$ for any $n \in \mathcal{N} \not\Rightarrow \Delta a$ P-improving. This can be illustrated in the utility space: a point on the frontier of the utility set (such as B in figure 6) corresponds to a zero-loss allocation whereas any interior point (such as A in figure 6) corresponds to a strictly positive loss. In figure 6, an allocation leading to B is not P-improving as compared to an allocation leading to A, and yet the loss is lower in B (zero) than in A.

The next proposition is due to Courtault and Tallon (2000).

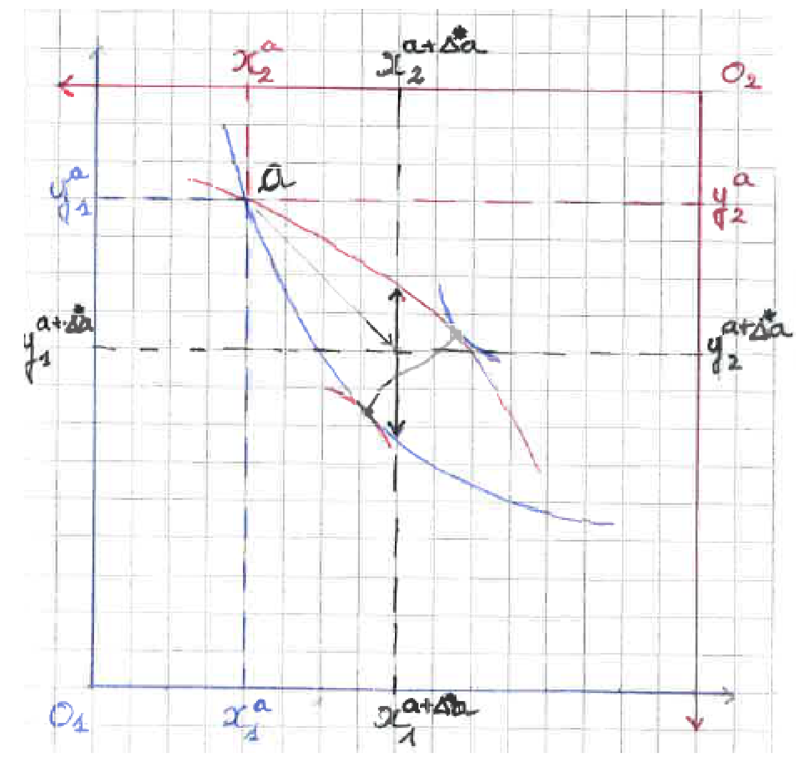

Figure 5: Surplus maximization and Pareto-efficiency.

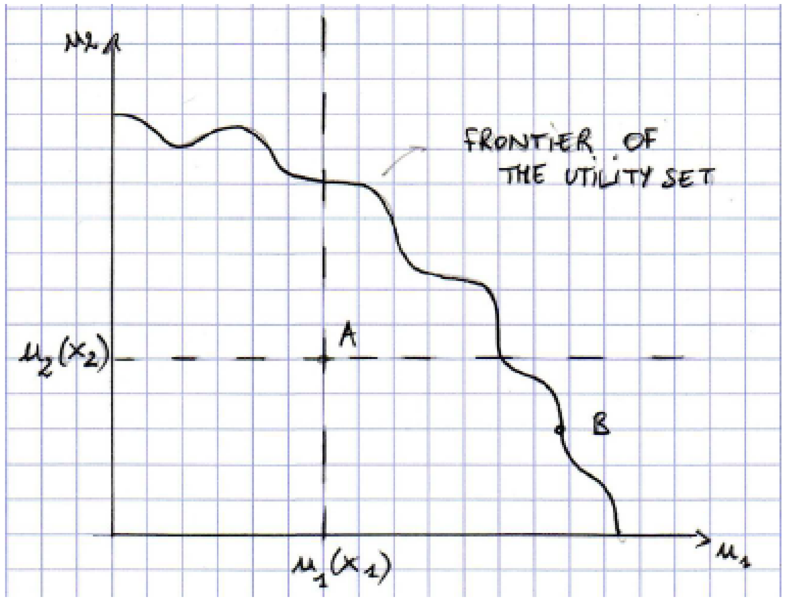

Figure 6: An allocation reducing loss is not necessary Pareto-improving.

Proposition 13 (Courtault and Tallon, 2000) *Suppose, for all $i \in \mathcal{I}$, $u_i(.)$ is continuous and strictly quasi-concave. From any $a \in \mathcal{A}$ and for any $n \in \mathcal{N}$, the sequence of reallocations $\{\Delta_t a\}_{t \in \mathbb{N}_+^*}$ defined, for all $t \in \mathbb{N}_+^*$, by*

$$\begin{aligned} \Delta_{t+1} a &= \arg \max_{\Delta a \in \mathcal{R}(a_t)} \Delta s_n (a_t) \\ a_t &= a_{t-1} + \Delta_t a \text{ and } a_0 = a, \end{aligned}$$

leads to a unique allocation $a^* = a + \sum_{t \in \mathbb{N}_+^*} \Delta_t a$ which is Pareto-efficient.

Proof. See Courtault and Tallon (2000). ■

Note that strict quasi-concavity of utility functions is not required in the TGS.

3 The TGS with "money"

The list of goods is extended with an additional one, a quantity of which is denoted y. Individual i's preferences are now defined over plans $(\mathbf{x}_i, y_i) \in \mathbb{R}^{N+1}$ and function $u_i(.)$ by $u_i = u_i(\mathbf{x}_i, y_i)$. So far, the proposed extension does not substantially change the analysis. To avoid any ambiguity as to the ability of some transformation unit to extract rent *for its own account*, one suggests not to follow Allais (TGS, part II, chapter 2) and to keep good Y away from transformation operation: it is assumed to be neither an input nor an output of any transformation process.[17] Since, individual themselves can only hold positive amounts of "money," previous restriction involves that "money" supply is inelastic. Furthermore, the only way to release surplus from a transformation operation is to reduce a preexisting "transformation loss."

Henceforth, an allocation writes $a = \left((\mathbf{x}_i, y_i)_{i \in \mathcal{I}}, (\mathbf{x}_j)_{j \in \mathcal{J}} \right)$ and it is *feasible* if and only if

$$\begin{aligned} \sum_{i \in \mathcal{I}} \mathbf{x}_i + \sum_{j \in \mathcal{J}} \mathbf{x}_j &\leq \overline{\mathbf{x}} \text{ and } \sum_{i \in \mathcal{I}} y_i \leq \overline{y}, \\ f_j(\mathbf{x}_j) &\geq 0 \text{ for all } j \in \mathcal{J}, \end{aligned}$$

where $(\overline{\mathbf{x}}, \overline{y})$ represent the preexisting global resources of the economy in each good; $\mathcal{A}$ still denotes the set of feasible allocation. A reallocation becomes a list of variations $\Delta a = \left((\Delta \mathbf{x}_i, \Delta y_i)_{i \in \mathcal{I}}, (\Delta \mathbf{x}_j)_{j \in \mathcal{J}} \right)$ and it is *feasible* if and only if

$$\begin{aligned} \sum_{i \in \mathcal{I}} \Delta \mathbf{x}_i + \sum_{j \in \mathcal{J}} \Delta \mathbf{x}_j &\leq \mathbf{0} \text{ and } \sum_{i \in \mathcal{I}} \Delta y_i \leq 0, \\ f_j(\mathbf{x}_j + \Delta \mathbf{x}_j) &\geq 0 \text{ for all } j \in \mathcal{J}. \end{aligned}$$

3.1 "Money"

The extra good Y, the purpose of which is to serve as a good of reference, is assumed to have specific properties. It is assumed to be *perfectly divisible*, as well as such that, for all $i \in \mathcal{I}$ and $\mathbf{x}_i \in \mathbb{R}^N$:

1. $u_i(\mathbf{x}_i, y_i)$ is *continuous* in y_i;
2. $u_i(\mathbf{x}_i, y_i)$ is *strictly increasing* in y_i;
3. $u_{iy}^m(\mathbf{x}_i, y_i) \underset{y_i \to 0}{\to} +\infty$ where $u_{iy}^m(\mathbf{x}_i, y_i)$ denotes the marginal utility of Y as measured at $(\mathbf{x}_i, y_i)$.

Allais comes to call "money" such a good of reference and allows it to be either a "commodity-money" (e.g. salt) or *fiat money*. Previous assumptions mean individuals are willing to hold "money" for itself. See the introduction for an extensive justification.

3.2 Surplus and loss as measured in "money"

Introducing a "money" greatly simplifies the analysis. It first allows to relax the assumption that $u_i(\mathbf{x}_i, y_i)$ is increasing in x_{in} for all $n \in \mathcal{N}$. In addition, it simplifies the definition of basic concepts.

[17] This is not a minor issue, as Currie and al. (1971) point out, since the surplus released by a production unit shall be accounted at the level of its owners.

3.2.1 Individual surplus

Definition 14 *For any individual $i \in \mathcal{I}$ with initial plan $(\mathbf{x}_i, y_i) \in \mathbb{R}^{N+1}$, the surplus Δs_i associated to the change $(\Delta \mathbf{x}_i, \Delta y_i)$ as measured in "money" is the amount $\Delta s_i \in \mathbb{R}$ such that $u_i(\mathbf{x}_i + \Delta \mathbf{x}_i, y_i + \Delta y_i - \Delta s_i) = u_i(\mathbf{x}_i, y_i)$.*

Starting from a...

- $\Delta s_i > 0$ is the maximal amount of "money" individual i would be willing to pay (willingness-to-pay) in exchange for the implementation of the change $(\Delta \mathbf{x}_i, \Delta y_i)$;

- $\Delta s_i < 0$ is the minimal amount of "money" individual i would call for (willingness-to-deliver) so as to accept the change $(\Delta \mathbf{x}_i, \Delta y_i)$.

Note that in the case where $\Delta \mathbf{x}_i\prime = (0, ..., 0, \Delta x_{in}, 0, ..., 0)$ with $\Delta x_{in} > 0$, and $\Delta y_i = 0$, the amount $\frac{\Delta s_i}{\Delta x_{in}}$ is simply the individual i (subjective) demand price for good n (inverse-demand). Furthermore, if $\Delta y_i = -p_n \Delta x_{in}$, where p_n is a given uniform price of good $n \in \mathcal{N}$, then Δs_i captures the standard Marshallian concept of surplus.

3.2.2 Collective surplus

Definition 15 *Given an initial allocation $a = \left((\mathbf{x}_i, y_i)_{i \in \mathcal{I}}, (\mathbf{x}_j)_{j \in \mathcal{J}}\right) \in \mathcal{A}$, the collective surplus Δs associated to a reallocation $\Delta a = \left((\Delta \mathbf{x}_i, \Delta y_i)_{i \in \mathcal{I}}, (\Delta \mathbf{x}_j)_{j \in \mathcal{J}}\right)$ as measured in "money" is the amount $\Delta s \in \mathbb{R}$ defined as*

$$\Delta s = \sum_{i \in \mathcal{I}} \Delta s_i - \sum_{i \in \mathcal{I}} \Delta y_i,$$

where, for all $i \in \mathcal{I}$: $u_i(\mathbf{x}_i + \Delta \mathbf{x}_i, y_i + \Delta y_i - \Delta s_i) = u_i(\mathbf{x}_i, y_i)$.

This definition involves that two reallocations giving the same quantity of each good to each individual but different amounts of money lead to the same surplus.

Claim 16 *Two reallocations which only differ with respect to "money" release the same surplus.*

Proof. See the appendix. ■

Corollary 17 *A reallocation which only changes individual cash balances releases no surplus at the scale of the economy.*

Releasing surplus is to provide the economy with a reallocation Δa desirable enough so that the total amount of "money" the "direct winners" are willing to pay to implement it exceeds what the "direct losers" call for in order to allow it. Once released, the surplus can be distributed or not.

3.2.3 Loss

Given $a \in \mathcal{A}$, let $l(a)$ denote the loss as measured in "money":

$$l(a) = \max_{\Delta a \text{ feasible}} \Delta s(a).$$

3.3 "Monetary" surplus and Pareto-efficiency

Money has the properties of a "natural" good of reference.

Proposition 18 *$l_n(a) = 0$ for all $n \in \mathcal{N} \Leftrightarrow l(a) = 0$.*

Proof. See the appendix. ∎

Useful to assess the concern of Debreu or Luenberger as to the choice of particular good of reference: with "money", this concern is pointless.

Proposition 19 *$a^* \in \mathcal{A}$ Pareto-efficient $\Leftrightarrow \Delta s(a^*) \leq 0$ for all feasible reallocation $\Delta a \Leftrightarrow l(a^*) = 0$.*

Proof. See the appendix. ∎

3.4 "Monetary" loss...

3.4.1 in the space of utility levels

Given $a \in \mathcal{A}$, denoting $u_i = u_i(\mathbf{x}_i, y_i)$, let's implicitly define $\Delta \mathfrak{s}_i(u_i; \Delta a)$ by

$$u_i(\mathbf{x}_i + \Delta \mathbf{x}_i, y_i + \Delta y_i - \Delta \mathfrak{s}_i(u_i; \Delta a)) = u_i$$

One can define function $\mathfrak{l}(u_1, ..., u_I; \overline{\mathbf{x}}, \overline{y})$ by

$$\mathfrak{l}(u_1, ..., u_I; \overline{\mathbf{x}}, \overline{y}) = \max_{\Delta a \text{ feasible}} \sum_{i \in \mathcal{I}} \Delta \mathfrak{s}_i(u_i; \Delta a) - \sum_{i \in \mathcal{I}} \Delta y_i$$

Claim 20 *For all $i \in \mathcal{I}$, the loss $\mathfrak{l}(u_1, ..., u_I; \overline{\mathbf{x}}, \overline{y})$ is decreasing with respect to u_i.*

Proof. See the appendix. ∎

The condition $\mathfrak{l}(u_1, ..., u_I; \overline{\mathbf{x}}, \overline{y}) = 0$ gives the equation of the *utility possibility frontier* in the space of utility levels. The shape of this frontier might be complex, however, given $(\overline{\mathbf{x}}, \overline{y})$, it is not possible that all utility levels increase together. In the space of utility levels, one can define an "iso-loss" set made of all the vectors of utility levels corresponding to a given loss.

Definition 21 *Given $\bar{l} > 0$, the condition $\mathfrak{l}(u_1, ..., u_I; \overline{\mathbf{x}}, \overline{y}) = \bar{l}$ gives the equation of the level $\bar{l}$* iso-loss set *in the space of utility levels.*

Constraints incompatible with the realization of a Pareto-efficient allocation can be dealt with: rather than Pareto-efficiency, one shall thus seek for a *least loss* allocation *i.e.* an allocation a solving $\min_{a \in \mathcal{A}} l(a)$ such that a meets further constraints specific to the problem under consideration. A second best setting.

3.4.2 as an index of inefficiency

Allais deems that, expressed in "money", his loss concept provides a correct quantitative index of inefficiency in the sense that it meets the following criteria (TGS, §547): (1) it depends only on the allocation under consideration / it is a clearly defined function of the utility levels and available resources; (2) it takes account of all structural conditions characterizing the economy but only of these conditions, excluding any given system of prices or any special system of organization; (3) it involves, on a symmetric basis, all the goods - except the good of reference ("money"), all the utility functions, and all the production functions; (4) it equals zero for all Pareto-efficient allocations; (5) it is positive for any P-inefficient allocations; (6) it decreases as a result of Pareto-improving reallocations; (7) its variations can be calculated for any subset of agents and for all allocations; (8) it remains the same following any increasing transformation of utility functions; (9) it is measured in terms of a good common to all agents ("money"); (10) it helps to analyze out-of-equilibrium dynamics as well as steady Pareto-efficient state; (11) it is independent from issue of income distribution; (12) it is independent from any restrictive conditions such as continuity (except as regards "money"), derivability or convexity.

4 A surplus-seeking economy

So far, surplus and loss are considered as tools for welfare analysis. Allais actually draws from these concepts a theory of the functioning of a free-trade economy. He makes his point through the notion of an "economy of markets" as opposed to the "market economy" (understood, "one-market economy") modeled by the standard walrasian theory. The expression "surplus-seeking economy" is preferred here. First, because the linguistic subtlety introduced by Allais is not particularly enlightening in English. Second, we believe that the expression "economy of markets" does not do full justice to the generality of the analysis proposed by Allais since, except as regards the principles of voluntary exchange and the right to property, no specific institutional setting is imposed in the analysis. Although he does not explicitly highlight this feature of the TGS, some of his remarks suggest he is aware of it (TGS, p. 362, §564).

4.1 Surplus-seeking as general form of economic behaviors

The TGS does not assume any pre-existing system of price; the number of individuals does not need to be high and perfect competition is not required (TGS, p. 362, §564). Economic interactions are not even necessarily intermediated by markets. Individuals have direct economic interactions (exchange or cooperation in production.) Allais' positive economics all stems from the statement that: "In essence any economic operation, whatever it may be, should be viewed as related to the search, realization and distribution of distributable surplus."[18]

[18] TGS, p. 32, §115, translation of Guesnerie (1984).

Individual surplus plays the central role.

4.1.1 Individual behaviors

Individuals in the TGS can be price-takers or not, opportunistic or not, well-informed or not, "full-appropriators" or not (Makowski and Ostroy, 2001). Their basic impetus is to search for one or more other individuals willing to accept bilateral or multilateral transactions (exchange or production) creating surplus that can be distributed (TGS, p. 360, §563). As production is considered, this involves that individuals seek technical efficiency.

4.1.2 Collective behavior

The TGS is primarily the theory of an *out-of-equilibrium* process of voluntary exchanges and cooperation (TGS, p. 427, §598). At any time, exchange and production operations occur at prices specific to the transactions involved (TGS, p. 334, §551,3). From the fact that, in such an economy, surpluses are constantly created and distributed, it follows that, while the utility of an individual grows, the utility levels of others can never decrease. This involves that, for given economic structures (preferences, technology, and resources), the process of voluntary exchanges and cooperation puts the economy closer and closer from a stable global P-efficient allocation (TGS, p. 360, §563). In general, no single transaction leads to a global P-efficient allocation but every such transaction makes this state closer.

4.2 General equilibrium as a "zero-releasable-surplus" allocation

In this perspective, an allocation is an equilibrium if and only if, no surplus can be released starting from this allocation. From previous analyses, it follows that an allocation is an equilibrium if and only if it is P-efficient (TGS, p. 335, §551,3). Since uniqueness is by no means a property of "equilibrium" understood in Allais' sense, it has no predictive purpose. Allais' point is to describe how out-of-equilibrium genuinely decentralized surplus-seeking behaviors may contribute to the realization of a P-efficient allocation. Individual incentives come from the prospect of a partial or full appropriation of released surplus. The fuel of this process is the *information* available as to surplus-releasing opportunities (TGS, p. 360, §563).

5 The marginal TGS

Allais' TGS is now considered assuming continuity and derivability in all dimensions, that is: for all $i \in \mathcal{I}$ and $n \in \mathcal{N}$, $u_i(.)$ derivable in x_{in}; for all $j \in \mathcal{J}$ and $n \in \mathcal{N}$, $f_j(.)$ derivable in x_{jn}. This allows to consider infinitesimal reallocations in the neighborhood of any allocation $a \in \mathcal{A}$ and to get linearized approximate expressions of surplus.

5.0.1

Under continuity and derivability assumptions, the model can be restated as follows.

- For all $i \in \mathcal{I}$, $u_i = u_i(\mathbf{x}_i, y_i)$ with: $u'_{in}(\mathbf{x}_i, y_i) \geq 0$ for all $n \in \mathcal{N}$; $u'_{iy}(\mathbf{x}_i, y_i) > 0$ and $\lim_{y_i \to 0} u'_{iy}(\mathbf{x}_i, y_i) \to +\infty$.[19]
- For all $j \in \mathcal{J}$, $f_j(\mathbf{x}_j) = 0$ with $f'_{jn}(\mathbf{x}_j) \geq 0$ for all $n \in \mathcal{N}$. This means that "productive efficiency" is no longer an issue hereafter. $f'_{jn}\left(= \frac{dl_j}{dx_{jn}}\right)$ is the marginal transformation loss attached to the use of the (resp. the production of) good n. *Producers' behavior consists in setting the transformation loss to zero i.e.* to produce efficiently.
- Use-resource balance: $\sum_{i \in \mathcal{I}} \mathbf{x}_i + \sum_{j \in \mathcal{J}} \mathbf{x}_j = \overline{\mathbf{x}}$ and $\sum_{i \in \mathcal{I}} y_i = \overline{y}$. These relations, most notably as regards the good of reference *i.e.* "money", describe a case of distributed surplus. To the extent that a non-satiation assumption is made, cases such that these relations do not hold are trivially Pareto inefficient.

Below, one will nonetheless consider cases of released non distributed surplus where $\sum_{i \in \mathcal{I}} y_i < \overline{y}$ and consider Pareto efficiency within an economy with a "money" supply $\sum_{i \in \mathcal{I}} y_i < \overline{y}$.

5.0.2

5.1 The TGS in terms of marginal valuations

Definition 22 *For all* $(\mathbf{x}_i, y_i) \in \mathbb{R}^{N+1}$*, let's define the* marginal subjective valuation function $v'_{in}(.)$ *as measured in "money" by*

$$v'_{in}(\mathbf{x}_i, y_i) = \frac{u'_{in}(\mathbf{x}_i, y_i)}{u'_{iy}(\mathbf{x}_i, y_i)}.$$

All other things being equal, starting from $(\mathbf{x}_i, y_i)$ and assuming $x_{in} > 0$, $v'_{in}(\mathbf{x}_i, y_i)$ is: (a) the *maximal* amount (of "money") individual i would be willing to pay in exchange for an additional unit of good (or service) n; (b) the *minimal* amount individual i would call for in exchange for a one unit reduction of his consumption of good n. If $x_{in} < 0$, $v'_{in}(\mathbf{x}_i, y_i)$ is: (a) the *maximal* amount individual i would be willing to give up to reduce by one unit the quantity of good (or service) n he delivers to the economy; (b) the *minimal* amount individual i would call for in exchange for a one unit increase in the quantity of good (or service) n he supplies to the economy.

[19] The argument here is that individuals (1) never believe for sure that they are in equilibrium, (2) value "money" because it gives access to markets.

5.1.1 The marginal surplus...

as measured in "money" By definition of individual surplus, for any list of infinitesimal variations $(d\mathbf{x}_i, dy_i)$:

$$ds_i = \sum_{n\in\mathcal{N}} v'_{in}(\mathbf{x}_i, y_i)\, dx_{in} + dy_i.$$

Starting from $(\mathbf{x}_i, y_i)$, the amount ds_i is the maximal contribution $(ds_i > 0)$ or the minimal compensation $(ds_i < 0)$ driving i to accept the individual change $(d\mathbf{x}_i, dy_i)$. Below, one considers some infinitesimal reallocation da (infinitesimal analogue of Δa) in the neighborhood of some allocation $a \in \mathcal{A}$.

Claim 23 *Given an initial allocation* $a = \left((\mathbf{x}_i, y_i)_{i\in\mathcal{I}}, (\mathbf{x}_j)_{j\in\mathcal{J}}\right) \in \mathcal{A}$, *the surplus associated to any infinitesimal reallocation* $da = \left((d\mathbf{x}_i, dy_i)_{i\in\mathcal{I}}, (d\mathbf{x}_j)_{j\in\mathcal{J}}\right)$ *can be written as*[20]

$$ds = \sum_{i\in\mathcal{I}}\sum_{n\in\mathcal{N}} v'_{in} dx_{in} = \sum_{i\in\mathcal{I}} (Dv_i)' d\mathbf{x}_i.$$

Proof. See the appendix. ■

The collective surplus associated to an infinitesimal reallocation directly follows from marginal subjective valuations.

The fact that surplus is insensitive to displacement of "money" is explicit in this expression. Further points deserve attention. First, due to the symmetry between goods (markets) and individuals, adding *market surplus* with respect to individuals or *individual surplus* with respect to markets is equivalent. It follows that, for infinitesimal variations, the marshallian partial analysis is adequate: total surplus is indeed the sum of the collective surpluses released on each market. Second, the writing $Dv_i(\mathbf{x}_i, y_i)$ refers to the gradient of some unobserved "total valuation function" which allows for cardinality. In the case where a "money" exists and differentiability can be assumed, surplus can duly be thought of in terms of valuation. Previous expression suggests an interesting link between the notion of "surplus release" and the informal one of "value creation".[21] Note that previous expression makes no assumption as to whether the reallocation da is feasible or not, whether surplus is retained or distributed.

Remark 24 *For any pair* $(\bar{\imath}, \hat{\imath})$ *of individuals, consider some good* n *for which* $v'_{\hat{\imath}n}(\mathbf{x}_{\hat{\imath}}^{\neg n}, x_{\hat{\imath}n}, y_{\hat{\imath}}) > v'_{\bar{\imath}n}(\mathbf{x}_{\bar{\imath}}^{\neg n}, x_{\bar{\imath}n}, y_{\bar{\imath}})$ *and suppose functions* $u_{\hat{\imath}}(\mathbf{x}_{\hat{\imath}}^{\neg n}, .)$ *and* $u_{\bar{\imath}}(\mathbf{x}_{\bar{\imath}}^{\neg n}, .)$ *are strictly quasi-concave. Then the reallocation, all other things being equal, of the good* n *between* $\hat{\imath}$ *and* $\bar{\imath}$ *defined by:*
(1) $(\Delta x_{\hat{\imath}n}, \Delta y_{\hat{\imath}}) = \left(\Delta x_n, \widehat{\Delta y}\right)$, $(\Delta x_{\bar{\imath}n}, \Delta y_{\bar{\imath}}) = \left(-\Delta x_n, \overline{\Delta y}\right)$ *with* $\Delta x_n > 0$, $\widehat{\Delta y} > 0$, $\overline{\Delta y} > 0$, *and*

[20] $(Dv_i)'$ denotes the transposed Dv_i gradient vector.
[21] Milgrom and Roberts (1992) use the expression in the quasi-linear case (no wealth effects).

(2) $u_{\hat{\imath}}\left(\mathbf{x}_{\hat{\imath}}^{\neg n}, x_{\hat{\imath}n}+\Delta x_n, y_{\hat{\imath}}-\widehat{\Delta y}\right)=u_{\hat{\imath}}\left(\mathbf{x}_{\hat{\imath}}, y_{\hat{\imath}}\right)$, $u_{\bar{\imath}}\left(\mathbf{x}_{\bar{\imath}}^{\neg n}, x_{\bar{\imath}n}-\Delta x_n, y_{\bar{\imath}}+\overline{\Delta y}\right)=u_{\bar{\imath}}\left(\mathbf{x}_{\bar{\imath}}, y_{\bar{\imath}}\right)$*, and*

$$v'_{\hat{\imath}n}\left(\mathbf{x}_{\hat{\imath}}^{\neg n}, x_{\hat{\imath}n}+\Delta x_n, y_{\hat{\imath}}-\widehat{\Delta y}\right)=v'_{\bar{\imath}n}\left(\mathbf{x}_{\bar{\imath}}^{\neg n}, x_{\bar{\imath}n}-\Delta x_n, y_{\bar{\imath}}+\overline{\Delta y}\right),$$

exists and releases a surplus $\widehat{\Delta y}-\overline{\Delta y}>0$ *which is the maximal releasable through a single transaction on* n *between* $\hat{\imath}$ *and* $\bar{\imath}$*.*

This is illustrated in figure 4.

as measured in labor Assume there exists one good $\underline{n}$ among all others, present in all utility functions (as a consumption $x_{i\underline{n}}>0$, or a delivery $x_{i\underline{n}}<0$), and required by all transformation operations (as an input, $x_{j\underline{n}}>0$); one can think of $\underline{n}$ as (unskilled) labor. Measuring surplus in labor allows to highlight a fundamental symmetry in the TGS. To see this, let $\mathcal{K}=\mathcal{I}\cup\mathcal{J}$ and define, for any $n\in\mathcal{N}-\{\underline{n}\}$:

$$\underline{v}'_{kn}\left(\mathbf{x}_k, y_k\right)=\begin{cases}\frac{u'_{kn}(\mathbf{x}_k,y_k)}{u'_{k\underline{n}}(\mathbf{x}_k,y_k)} & \text{if } k\in\mathcal{I}\\ \frac{f'_{kn}(\mathbf{x}_k)}{f'_{k\underline{n}}(\mathbf{x}_k)} & \text{if } k\in\mathcal{J}\end{cases},$$

and $\underline{v}'_{ky}\left(\mathbf{x}_k, y_k\right)=\frac{u'_{ky}(\mathbf{x}_k,y_k)}{u'_{k\underline{n}}(\mathbf{x}_k,y_k)}$ for $k\in\mathcal{I}$. A quantity $\underline{v}'$ expresses in units of labor; it may represent a marginal rate of substitution, a technical marginal rate of substitution, or a marginal rate of transformation, depending on the signs of x_{kn} and $x_{k\underline{n}}$. Consider $\left(k,\hat{k}\right)\in\mathcal{K}\times\mathcal{K}$, $\hat{k}\neq k$ and let $d_{\hat{k}}x_{kn}$ denote a (net) transfer of good n from $\hat{k}$ to k. One gets: $dx_{kn}=\sum_{\hat{k}\neq k}d_{\hat{k}}x_{kn}$ and $d_k x_{\hat{k}n}=-d_{\hat{k}}x_{kn}$. The marginal surplus as measured in labor admits an expression which comes to consider the economy at the scale of the *transaction* rather than at the scale of the individual or market. It further allows to explicit the role of transformation units in the process of surplus release.

Proposition 25 *Consider* $a\in\mathcal{A}$ *such that: for all* $i\in\mathcal{I}$, $u'_{i\underline{n}}\left(\mathbf{x}_i, y_i\right)>0$ *and, for all* $j\in\mathcal{J}$, $f'_{j\underline{n}}\left(\mathbf{x}_j\right)>0$*. The surplus, as measured in labor, associated to any infinitesimal reallocation* da*, can be written*

$$d\underline{s}=\sum_{n\neq\underline{n}}\sum_{k\in\mathcal{K}}\sum_{\hat{k}>k}\left(\underline{v}'_{kn}-\underline{v}'_{\hat{k}n}\right)d_{\hat{k}}x_{kn}+\sum_{i\in\mathcal{I}}\sum_{\hat{\imath}>i}\left(\underline{v}'_{iy}-\underline{v}'_{\hat{\imath}y}\right)d_{\hat{\imath}}y_i-d\underline{l}_{\mathcal{J}}.$$

Proof. See the appendix. ■

This writing highlights three ways to release surplus: (1) transactions between individuals;[22] (2) transactions between transformation units;[23] (3) trans-

[22] Between two individuals as consumers, between two individual producers, or between a consumer and an individual producer.

[23] Between two transformation units as intermediate consumers, between two producers, or between a producer and an intermediate consumer.

actions between a transformation unit and an individual.[24] It indicates that the possibility to release surplus may come from: differences between marginal valuations $v'_{in} - v'_{\hat{\imath}n}$, differences between marginal transformation losses $f'_{jn} - f'_{\hat{\jmath}n}$, or differences between some marginal valuation and marginal transformation loss, $v'_{in} - f'_{jn}$.

5.1.2 Variations of the marginal surplus

Let the list $\left(\left(d^2\mathbf{x}_i, d^2y_i\right)_{i\in\mathcal{I}}, \left(d^2\mathbf{x}_j\right)_{j\in\mathcal{J}}\right)$ captures the tendency of a reallocation da. For any pair $(k,n) \in \mathcal{K} \times \mathcal{N}$, four types of reallocations ought to be distinguished.

		d^2x_{kn}	
		> 0	< 0
dx_{kn}	> 0	positive increasing	negative decreasing
	< 0	positive decreasing	negative increasing

The point here is to be able to know whether the direction taken by a reallocation induces increasing or decreasing *returns in surplus i.e.* whether a further reallocation in the same direction might indeed increase surplus or not. The information required to answer is contained in the variations of the marginal subjective valuations associated to the initial allocation $a \in \mathcal{A}$.

Infinitesimal variations of the individual surplus

Claim 26 *Given a list* $\left(\left(d\mathbf{x}_i, dy_i\right), \left(d^2\mathbf{x}_i, d^2y_i\right)\right)_{i\in\mathcal{I}}$ *of infinitesimal changes in individual i's situation, variations of surplus can be written*

$$d^2s_i = (d\mathbf{x}_i)' \, D^2v_i\big|_{du_i=0} \, d\mathbf{x}_i + (Dv_i)' d^2\mathbf{x}_i + d^2y_i,$$

with $D^2v_i\big|_{du_i=0} = \left(v''_{in\bar{n}} - v'_{i\bar{n}}v''_{iny}\right)_{(n,\bar{n})\in\mathcal{N}^2}$ *and* $Dv_i = (v'_{in})_{n\in\mathcal{N}}$.

Proof. See the appendix. ■

Under continuity and derivability assumptions, the impact of an infinitesimal change $(d\mathbf{x}_i, dy_i)$ on individual i's valuation of good n is $dv'_{in} = \sum_{\bar{n}\in\mathcal{N}} v''_{in\bar{n}}dx_{i\bar{n}} + v''_{iny}dy_i$. Suppose that the net variation in "cash balances" exactly compensates the impact of $d\mathbf{x}_i$ on individual i's welfare *i.e.* $dy_i = -\sum_{\bar{n}\in\mathcal{N}} v'_{i\bar{n}}dx_{i\bar{n}}$. Corresponding variation in individual i's valuation of good n can be written:

$$dv'_{in}|_{du_i=0} = \sum_{\bar{n}\in\mathcal{N}} v''_{in\bar{n}}dx_{i\bar{n}} + v''_{iny}\cdot\left(-\sum_{\bar{n}\in\mathcal{N}} v'_{i\bar{n}}dx_{i\bar{n}}\right) = \sum_{\bar{n}\in\mathcal{N}} \left(v''_{in\bar{n}} - v''_{iny}v'_{i\bar{n}}\right) dx_{i\bar{n}}.$$

[24] Between a producer and a consumer, between a transformation unit and an individual producer, between a consumer and a transformation unit as an intermediate consumer.

Given an initial individual situation $(\mathbf{x}_i, y_i)$, let $dv'_{in}|_{du_i=0}$ denote a variation in individual i's valuation of good n as resulting from a change $d\mathbf{x}_i$, assuming dy_i exactly compensates the impact of $d\mathbf{x}_i$ on individual i's welfare. It captures the *substitution/complement effects* of $d\mathbf{x}_i^{\neg n}$ on v'_{in}, as well as the direct effect of dx_{in}, net of *welfare effects*.[25] This is a "compensated price demand", which accommodates the hicksian partial analysis concept "compensated demand" and the marshallian "demand price" within a general interdependence framework. The wealth effect is all absorbed in surplus.

In general, the change $(d\mathbf{x}_i, dy_i)$ induces a deformation of the whole individual i's *system of valuations*.

Infinitesimal variation of the collective surplus Considering the economy as whole, and a list of infinitesimal variations $\left(\left(\left(d\mathbf{x}_i, d^2\mathbf{x}_i\right), \left(dy_i, d^2y_i\right)\right)_{i\in\mathcal{I}}, \left(d\mathbf{x}_j, d^2\mathbf{x}_j\right)_{j\in\mathcal{J}}\right)$, a variation of collective surplus is simply $d^2s = \sum_{i\in\mathcal{I}} d^2s_i - \sum_{i\in\mathcal{I}} d^2y_i$ so that

$$d^2s = \sum_{i\in\mathcal{I}} \left((d\mathbf{x}_i)' \, D^2v_i\big|_{du_i=0} \, d\mathbf{x}_i + (Dv_i)' d^2\mathbf{x}_i \right).$$

The first term captures the variation in surplus as resulting from the deformation of marginal valuations due to the reallocation. The second term captures the variation in surplus as resulting from the tendency of the reallocation for given initial marginal valuations.

Note that da is not necessarily feasible. It is just a linear function of the infinitesimal variations.

5.2 Pareto efficiency

Necessary and sufficient condition for an allocation to realize a Pareto-efficient allocation; tangential solution.

Definition 27 *A reallocation Δa is* reversible *if, at the second order approximation,*[26] *the reallocation $-\Delta a$ is feasible as well.*[27]

Proposition 28 *$a \in \mathcal{A}$ Pareto-efficient $\Leftrightarrow$ $ds(a) = 0$ and $d^2s(a) \leq 0$ for all feasible (and reversible) reallocations da.*

Proof. See the appendix. ■

[25] One obviously has: $D^2v_i\big|_{du_i=0} = \sum_{n\in\mathcal{N}} dv'_{in}\big|_{du_i=0} \, dx_{in}$.

[26] That is, not taking account of the tendency of the reallocation.

[27] The notion is borrowed from thermodynamics principles: a transformation (in the sense admitted in physics) of a system is said to be reversible if it leaves constant the entropy.

5.2.1 Pareto-efficiency (tangential) necessary condition

If $a \in \mathcal{A}$ Pareto-efficient then $ds(a) = 0$: one now gives an economic interpretation of that condition. It is shown that if an allocation is Pareto-efficient: (1) valuations by all individuals for a given good are all aligned (*i.e.* individuals all agree on the "monetary" value of each good in the economy); (2) for any pair of different goods, (technical) marginal rates of substitution (or transformation) are aligned whatever the individuals or transformation technologies under consideration - remind that for all $(k,n) \in \mathcal{K} \times \mathcal{N}$, $x_{kn} \gtreqless 0$.

Proposition 29 *Given an allocation $a \in \mathcal{A}$, $ds(a) = 0$ if and only if*
(1) for any pair of individuals $(i, \bar{\imath}) \in \mathcal{I}^2$, $\bar{\imath} \neq i$, and any good $n \in \mathcal{N}$:

$$v'_{in}\big|_a = v'_{\bar{\imath}n}\big|_a$$

(2) for any pair of agents $(i,j) \in \mathcal{I} \times \mathcal{J}$, and any pair of goods $(n, \bar{n}) \in \mathcal{N}^2$, $\bar{n} \neq n$:

$$\left.\frac{f'_{jn}}{f'_{j\bar{n}}}\right|_a = \left.\frac{v'_{in}}{v'_{i\bar{n}}}\right|_a = \left.\frac{u'_{in}}{u'_{i\bar{n}}}\right|_a .$$

Proof. See the appendix. ∎

5.2.2 Pareto-efficiency second order condition

The expression of $d^2 s$ is considered starting from an allocation such that $ds = 0$ for all feasible reallocation.

Claim 30 *If $a \in \mathcal{A}$ is such that $ds(a) = 0$ and some good $\underline{n} \in \mathcal{N}$ exists such that $f'_{j\underline{n}} > 0$ for all $j \in \mathcal{J}$ (labor) then:*

$$d^2 s = \sum_{i \in \mathcal{I}} (d\mathbf{x}_i)' \, D^2 v_i\big|_{du_i = 0} \, d\mathbf{x}_i + \sum_{j \in \mathcal{J}} (d\mathbf{x}_j)' \, D^2 v_j\big|_{dl_j = 0} \, d\mathbf{x}_j$$

where, for all $j \in \mathcal{J}$: $D^2 v_j = \frac{p_{\underline{n}}}{f'_{j\underline{n}}} D^2 f_j$.

Proof. See the appendix. ∎

Previous expression results from the fact that $ds = 0$ involves that all (technical) marginal rates of substitution (of transformation) are aligned. Provided that $ds = 0$, ds^2 is thus a sum of quadratic forms in $\left((d\mathbf{x}_i)_{i \in \mathcal{I}}, (d\mathbf{x}_j)_{j \in \mathcal{J}}\right)$. "Monetary" variations $(dy_i)_{i \in \mathcal{I}}$ have no part; neither do the tendency of the reallocation as captured by $\left(\left(d^2\mathbf{x}_i, d^2 y_i\right)_{i \in \mathcal{I}}, \left(d^2\mathbf{x}_j\right)_{j \in \mathcal{J}}\right)$.

The next proposition states second order condition for an allocation to be Pareto-efficient.

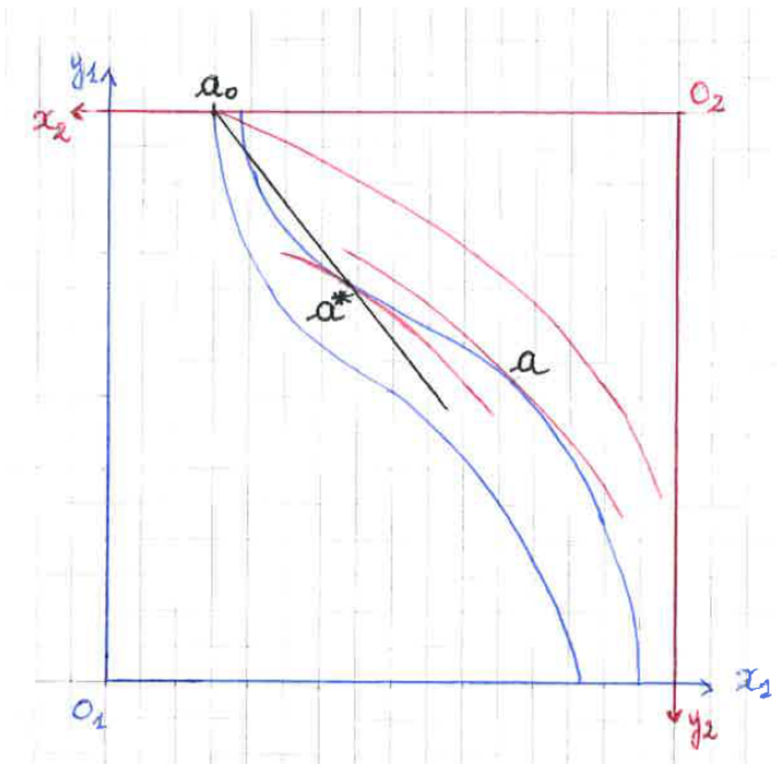

Figure 7: Necessary and sufficient condition for P-efficiency without global convexity

Proposition 31 *Given a feasible allocation* $a = \left((\mathbf{x}_i, y_i)_{i\in\mathcal{I}}, (\mathbf{x}_j)_{j\in\mathcal{J}}\right) \in \mathcal{A}$ *such that* $ds\,(a) = 0$ *for all feasible and reversible reallocations if*

$$\sum_{i\in\mathcal{I}} (d\mathbf{x}_i)' \, D^2 v_i\big|_{du_i=0} \, d\mathbf{x}_i + \sum_{j\in\mathcal{J}} (d\mathbf{x}_j)' \, D^2 v_j\big|_{dl_j=0} \, d\mathbf{x}_j \leq 0$$

for all feasible reallocations, then a is Pareto-efficient.

To be P-efficient, an allocation such that $ds = 0$ for all reallocations, must further exhibit *decreasing marginal returns* in surplus for all reallocations. At the scale of decision units (individuals or transformation units), at most one can be in a situation of local concavity: in general, previous inequality states that all agents exhibit *local* convexity *i.e.* decreasing marginal utility for individuals, decreasing marginal return for transformation units.

Example 32 *The case in figure 7:*
$v_1'\big|_a = v_2'\big|_a$ *but* $\left(v_{1x}''\big|_a - v_1'\big|_a \, v_{1y}''\big|_a\right) + \left(v_{2x}''\big|_a - v_2'\big|_a \, v_{2y}''\big|_a\right) > 0;$

$$v_1'|_{a^*} = v_2'|_{a^*} \text{ and } \left(v_{1x}''|_{a^*} - v_1'|_{a^*}\, v_{1y}''\big|_{a^*}\right) + \left(v_{2x}''|_{a^*} - v_2'|_{a^*}\, v_{2y}''\big|_{a^*}\right) < 0.$$

5.3 The walrasian general equilibrium and the TGS

5.3.1 Allais' criticism of the walrasian concept of general equilibrium

Allais' objections against the walrasian theory of general equilibrium are gathered in the paragraph §558 (TGS, p. 345). They can be summarized as follows.

1. Assumptions of the walrasian theory are inconsistent with observation
   The model relies on virtual behaviors (notional demand and supply functions) of agents facing an environment where: a single common system of prices exists, possible trading partners remain unspecified, supplies and demands confront with no actual exchanges, all transactions allowing to realize equilibrium occur in one step, once an equilibrium system of prices has been found. Allais' point is that: (1.1) there is never a single common system of prices; (1.2) there is no centralized marketplace.

2. The walrasian concept of general equilibrium is incompatible with any non-convexity, either in consumption or in production.

The interesting thing is that these objections have to do with a lack of realism: Allais' criticism targets a positive interpretation of the walrasian concept of general equilibrium.

One modern way to justify a positive interpretation of the WGE lies on the concept of *core* in cooperative game theory. Within an exchange economy (or a production economy with constant returns to scale), when the number of individuals tends to infinity, the allocation corresponding to a WGE is the only P-efficient allocation exhibiting the *core property* that is, such that there exists no coalition of individuals able to improve (for their own account) upon it (see Mas-Collel, 1995, p. 654-657).[28] Allais (TGS, §560, p. 356) raises again the issue of a lack of realism, pointing out the assumptions of general convexity and ever possible recontracting (or perfect information).

5.3.2 Thinking about the walrasian concept of general equilibrium from the TGS

Since it allows for out-of-equilibrium transactions, a "zero-releasable surplus" general equilibrium allocation is generally not a WGE: the former does not requires that the displacement from the initial allocation to the equilibrium takes the shape of a straight line (see Figure 7). But since any "zero-releasable surplus" general equilibrium is a P-efficient allocation, it induces a single common price system.

Corollary 33 *$a \in \mathcal{A}$ Pareto-efficient $\Rightarrow$ a common "monetary" price vector $(p_n)_{n\in\mathcal{N}}$ exists defined for all $n \in \mathcal{N}$ and pairs $(i,\bar{\imath}) \in \mathcal{I}^2$, $i \neq \bar{\imath}$, by $p_n = v_{in}' = v_{\bar{\imath}n}'$.*

[28] Theorem of Scarf.

Whereas the WGE concept is built on the (out-of-equilibrium) uniqueness of the price system (assumption grounded on perfect competition) and proves to correspond to a Pareto-efficient allocation, the TGS considers Pareto-efficiency as the criterion for an allocation to be an economic equilibrium and notes that it implies a common price system. Provided one deals with allocations satisfying supply-use balance for each good, the Pareto efficiency of walrasian equilibrium allocations (first theorem of welfare economics) directly follows from the assumption of a single price system common to all agents. It guarantees that individuals coordinate on an efficient use of resources. Restricting the analysis to allocations consistent with a common system of prices *de facto* amounts to restrict the analysis to Pareto-efficient allocations.

Global convexity for all individuals and technologies induces $(d\mathbf{x}_i)' \, D^2 v_i\big|_{du_i=0} (\mathbf{x}_i) \, d\mathbf{x}_i \leq 0$ and $\sum_{j \in \mathcal{J}} (d\mathbf{x}_j)' \, D^2 v_j\big|_{dl_j=0} d\mathbf{x}_j \leq 0$ for all $\left((\mathbf{x}_i)_{i \in \mathcal{I}}, (\mathbf{x}_j)_{j \in \mathcal{J}} \right)$ in which case $ds = 0$ is a necessary and sufficient condition for an allocation to the P-efficient. In such a case, any WGE corresponds to a Pareto-efficient allocation and any Pareto-efficient allocation can be generated as a WGE (setting the correct system of prices).

6 Conclusion

The WGE theory keeps a central role as regards research and teaching in economics: a reference in organizing economic thought; an analytical basis to a multitude of developments. But it is also at the heart of skepticism aroused by economic theory: interrogation as to what it represents (normative theory of value or model of a market economy); occultation of the fundamental issues of markets functioning and price formation; support to ideological reasonings (celebration of competition over cooperation). The TGS provides an analytical infrastructure useful both to unify microeconomic theory, and to think about economics with an open mind.

> Par sa nature même, en mettant l'accent sur la réalisation des surplus, le modèle de l'économie de marchés est essentiellement **dynamique** [...] Le modèle d'une économie de marchés centre son analyse sur les enchaînements de causalité, la recherche de surplus réalisables et leur réalisation constituant le principe fondamental et synthétique du fonctionnement de toute économie. (Economie de marchés et économie de marché, TGS, p. 362, §564)

6.1 The TGS as a unifying analytical infrastructure

The TGS helps connecting several important topics and schools of thought from microeconomic theory.

> Cette analyse [...] inclut comme autant de cas particuliers l'approche de Walras et Pareto, l'approche d'Edgeworth du "recontract", l'approche

marginaliste, l'approche marshallienne des économies et des coûts externes, la théorie des coalitions, et toutes les approches contemporaines. Elle permet de démontrer très facilement et rigoureusement toutes les propositions fondamentales. (Vue d'ensemble, TGS, §565, p. 364).

Let's focus on two important aspects.

6.1.1 The TGS and Marshall

Allais meets Marshall by considering a "monetary" economy, by providing an "out-of-equilibrium" analysis, and by rehabilitating partial equilibrium analysis understood as a local equilibrium analysis.

> [La TGS] retient les caractères essentiels de la réalité. (a) Il n'y a pas, **hors équilibre**, un système de prix unique pour tous les opérateurs, mais des systèmes de prix spécifiques à chaque opération d'échange. (b) Il n'y a pas de marché général et centralisé pour tous les biens, mais un ensemble de **marchés partiels**, chaque marché se rapportant à l'échange d'un seul bien contre un bien commun à tous les opérateurs [la "monnaie"] et n'étant pas nécessairement le seul où ce bien est échangé. (c) Sur chaque **marché partiel** se fixe un prix par confrontation des offres et des demandes, et la fixation de ce prix et suivi d'échanges effectifs. (d) Les échanges ont généralement lieu entre des opérateurs définis et des prix spécifiques à ces opérateurs. (e) Un marché peut se réduire à la rencontre de deux opérateurs pour une transaction à un prix défini. (f) Les échanges (et les décisions de production correspondantes) ne sont pas effectués en une seule fois en utilisant un seul système de prix, et **l'évolution de l'économie** vers l'équilibre se fait à la suite d'échanges successifs (et des opérations de production correspondantes) au cours de périodes successives utilisant des systèmes de prix différents. (Le modèle de l'économie de marché et la réalité, TGS, p.359, §562)
>
> Le processus d'évolution d'une économie de marchés consiste dans une suite d'**équilibres** successifs sur des marchés **partiels**. Si on suppose que sa structure reste la même au cours du temps et si ses principes dynamiques sont observés, on aboutit nécessairement à une situation d'équilibre qui est également d'efficacité maximale [...]. (Economie de marchés et économie de marché, TGS, p. 362, §564)

Caring about out-of-equilibrium economics (as Marshall with the concept of temporary market equilibrium) does justice to Keynes' concern with short-period analysis.

6.1.2 The TGS and Coase

As suggested by Guesnerie (1984), there are remarkable meeting points between Allais' model and a still in-progress coasian model of the economy lying on Pareto-efficiency.

> Although Allais does not seem to be totally explicit about whether, in the spirit of the Coase theorem, [...] surplus would actually be totaly exhausted through economic activity, he emphasizes the role of [...] surplus as the analogue of 'potential" in physics, or [...] as the natural 'Lyapounov function' of non-tâtonnement theory." (Guesnerie, 1984, p. 782)

In his Nobel address, Coase observes that

> [...] the concept of transaction costs has not been incorporated into a general theory. [...] incorporating transaction costs into standard economic theory, which has been based on the assumption that they are zero, would be very difficult, and economists [...] have not been inclined to attempt it. (Coase, 1992, p. 718)

Understanding transaction costs Quotation from "The nature of the firm":

> The most obvious cost of "organizing" production through the price mechanism is that of discovering what the relevant prices are. (Coase, 1937, p. 390)

Very difficult to see what Coase is referring to within the walrasian framework, it becomes much clearer within the TGS. TGS well suited to deal with transaction costs: transaction-based analysis, imperfect information, out-of-equilibrium analysis.

> I know of only one part of economics in which transaction costs have been used to explain a major feature of the economic system, [...] the evolution and use of money. (Coase, 1992, p. 716)

Money solves some transactional problems facilitating some multilateral transactions by changing them into a list of bilateral transactions. But some multilateral transactions remain to be achieved and are impeached by imperfect information or strategic opportunistic behaviors.

Thinking about the "Coase Theorem" Quotations

> La TGS montre que les purs transferts de revenus sont sans effet sur l'efficacité de l'économie. (Transferts de revenus et efficacité de l'économie, TGS, p. 427, §598)

La TGS permet de prendre facilement en compte les effets externe. (Prise en compte des effets externes dans le calcul économique, TGS, p. 429, §598)

La considération des surplus permet de tenir compte facilement des coûts externes, soit qu'on en tienne directement compte dans les indices de préférence, soit qu'on les déduise des surplus bruts qu'un calcul économique relatif à une décision peut faire apparaître. (Coût externe, TGS, p. 152, §324)

6.2 A framework which opens minds

A stepping stone for useful fresh developments in microeconomic theory.

6.2.1 Economic organizing principles beyond competition

Interestingly, this analysis is not grounded on a concept of competition. The main focus is on exchange and cooperation which translates into the use of Pareto efficiency as the main concept. Surplus allows to accomodate some space between general equilibrium and cooperative game theory.

6.2.2 Relational economic behaviors

The TGS helps to move microeconomics from a solipsistic view of economic behaviors (each individual anonymous and facing a price system) to a relational one. Individuals in Allais' analysis have relationships: they are social beings in the sense that the collect information about others seeking for surplus-releasing opportunities. They do have direct interactions with one another that can take many forms beyond market transactions, the outcome of which is the realization of surplus. Much easier to connect economics to social sciences within such a framework. Nevertheless, the general interdependence of transactions is properly accounted for.

6.2.3 Thinking about what microeconomic policy is about

Microeconomic policy cannot be reduced to "structural adjustment" *i.e.* the idea that it is all about boosting competition on markets. Rather, it consists of identifying transaction obstacles (imperfect information and/or strategic uncertainty) and the ways to remove them in order to release surplus. It basically sustains the mechanism design agenda on a infra-analytical basis (no specification of information structure): market is one tool facilitating transactions, firm is another, and still further consciously designed coordination mechanisms deserve attention.

6.2.4 Teaching microeconomic theory

Undergraduate textbooks typically start by setting efficient allocation of scarce resources as the fundamental issue of economics; it is then argued that a system of competitive markets provides a satisfying response (with usual reserves). The next chapters are devoted to the gradual elaboration of the theory of WGE, supposed to contain a formal proof of what competitive markets acheive. The trouble experienced by students is that, at the end, coordination does not seem to be realized through a decentralized process of supply and demand adjustments but rather by a central planner (the walrasian auctioneer)... who was not even mentioned in the first place. To conceal this disconcerting contradiction, the trick is to insert between the analysis of individual price-taking behaviors and the WGE model, some partial equilibrium analyses (of marshallian nature); with a little skill, it is enough to convince students that the treatment is all consistent. Because, the TGS is in line with the marshallian approach, it brings a more satisfying presentation of what markets are expected to achieve (be they competitive or not). The presentation of the theory of WGE could thus be reserved to chapters devoted to mechanism design (auctions) or to normative economics (with the notion that, at the WGE, each individual gets its marginal contribution to the economy).

The TGS is more generally well suited to deal with the issues of teaching economics. Proposition 25 is a good illustration of this aspect: it sums up the whole marginalist's theory in one equation. More importantly, the TGS allows a position of exteriority as regards markets and helps understanding the institutionalist's point: that economics is about making mutually advantageous transactions possible!

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A The TGS

Proof 4. For all $i \in \mathcal{I}$, consider the function $U_i(.)$ defined for all $\mathbf{x}_i \in \mathbb{R}^N$ by $U_i(\mathbf{x}_i) = T_i(u_i(\mathbf{x}_i))$ where $T_i(.)$ is a monotonous, strictly increasing function of u_i. For any $\bar{n} \in \mathcal{N}$, let $\Delta S_{i\bar{n}}$ denote the surplus corresponding to a given reallocation: $\Delta S_{i\bar{n}} = \sum\limits_{i\in\mathcal{I}} \Delta S_{i\bar{n}} - \sum\limits_{i\in\mathcal{I}} \Delta x_{i\bar{n}}$ where, for all $i \in \mathcal{I} : \Delta S_{i\bar{n}} = \max\{\Delta\sigma_{i\bar{n}} \in \mathbb{R} \mid U_i(\mathbf{x}_i^{\neg\bar{n}} + \Delta\mathbf{x}_i^{\neg\bar{n}}, x_{i\bar{n}} + \Delta x_{i\bar{n}} - \Delta\sigma_{i\bar{n}}) \geq U_i(\mathbf{x}_i)\}$. Clearly, for all $i \in \mathcal{I}$:

$$
\begin{aligned}
& \max\left\{\Delta\sigma_{i\bar{n}} \in \mathbb{R} \mid U_i\left(\mathbf{x}_i^{\neg\bar{n}} + \Delta\mathbf{x}_i^{\neg\bar{n}}, x_{i\bar{n}} + \Delta x_{i\bar{n}} - \Delta\sigma_{i\bar{n}}\right) \geq U_i(\mathbf{x}_i)\right\} \\
= \; & \max\left\{\Delta\sigma_{i\bar{n}} \in \mathbb{R} \mid T_i\left(u_i\left(\mathbf{x}_i^{\neg\bar{n}} + \Delta\mathbf{x}_i^{\neg\bar{n}}, x_{i\bar{n}} + \Delta x_{i\bar{n}} - \Delta\sigma_{i\bar{n}}\right)\right) \geq T_i(u_i(\mathbf{x}_i))\right\} \\
= \; & \max\left\{\Delta\sigma_{i\bar{n}} \in \mathbb{R} \mid u_i\left(\mathbf{x}_i^{\neg\bar{n}} + \Delta\mathbf{x}_i^{\neg\bar{n}}, x_{i\bar{n}} + \Delta x_{i\bar{n}} - \Delta\sigma_{i\bar{n}}\right) \geq u_i(\mathbf{x}_i)\right\} \\
= \; & \Delta s_{i\bar{n}},
\end{aligned}
$$

and $\Delta S_{i\bar{n}} = \sum\limits_{i\in\mathcal{I}} \Delta s_{i\bar{n}} - \sum\limits_{i\in\mathcal{I}} \Delta x_{i\bar{n}} = \Delta s_{i\bar{n}}$. ■

Proof 12. Given $\Delta a = \left((\Delta\mathbf{x}_i)_{i\in\mathcal{I}}, (\Delta\mathbf{x}_j)_{j\in\mathcal{J}}\right)$, for any $n \in \mathcal{N}$, consider the list $(\Delta s_{in})_{i\in\mathcal{I}}$ as defined by

$$\Delta s_{in} = \max\{\Delta\sigma_{in} \in \mathbb{R} \mid u_i(\mathbf{x}_i^{\neg n} + \Delta\mathbf{x}_i^{\neg n}, x_{in} + \Delta x_{in} - \Delta\sigma_{in}) \geq u_i(\mathbf{x}_i)\}$$

for all $i \in \mathcal{I}$. Since Δa is P-improving, $\Delta s_{in} \geq 0$ for all $i \in \mathcal{I}$ with at least one strict inequality. It follows that the surplus released by Δa is $\Delta s_n(a) = \sum\limits_{i\in\mathcal{I}} \Delta s_{in} > 0$. The maximal releasable surplus from $a_+ \equiv a + \Delta a$ is $l_n(a_+)$ by definition; let $\Delta^* a_+$ denote the reallocation corresponding to the releasing of $l_n(a_+)$. The maximal surplus released from a through the pair of reallocations $(\Delta a, \Delta^* a_+)$ is $\Delta s_n(a) + l_n(a_+)$. By definition, the maximal surplus releasable from a is $l_n(a)$ which involves $\Delta s_n(a) + l_n(a_+) \leq l_n(a)$. Since $\Delta s_n(a) > 0$, one concludes that $l_n(a_+) \leq l_n(a)$. ■

Proof 8. Two steps.
(1) a^* P-efficient $\Rightarrow l_n(a^*) = 0, \forall n \in \mathcal{N}$. This restates: $\exists n \in \mathcal{N}$ s.t. $l_n(a^*) > 0 \Rightarrow a^*$ not P-efficient. $\exists n \in \mathcal{N}$ s.t. $l_n(a^*) > 0 \Rightarrow \exists n \in \mathcal{N}$ and $\Delta a^* = \left((\Delta\mathbf{x}_i^*)_{i\in\mathcal{I}}, (\Delta\mathbf{x}_j^*)_{j\in\mathcal{J}}\right)$ feasible such that $\Delta s_n(a^*) > 0$. Consider the allocation $\left(\left(\mathbf{x}_i^{\neg n*} + \Delta\mathbf{x}_i^{\neg n*}, x_{in}^* + \Delta x_{in}^* - \Delta s_{in} + \frac{\Delta s_n(a^*)}{I}\right)_{i\in\mathcal{I}}, \left(\mathbf{x}_j^* + \Delta\mathbf{x}_j^*\right)_{j\in\mathcal{J}}\right)$, it is both feasible and such that $\forall i \in \mathcal{I}$:

$$u_i\left(\mathbf{x}_i^{\neg n*} + \Delta\mathbf{x}_i^{\neg n*}, x_{in}^* + \Delta x_{in}^* - \Delta s_{in} + \frac{\Delta s_n(a^*)}{I}\right) \geq u_i(\mathbf{x}_i^{\neg n*} + \Delta\mathbf{x}_i^{\neg n*}, x_{in}^* + \Delta x_{in}^* - \Delta s_{in}) \geq u_i(\mathbf{x}_i^*),$$

by definition of $\Delta s_n(a^*)$. Furthermore, preferences for good n are non-satiated for at least one individual, $\exists i \in \mathcal{I}$ for whom

$$u_i\left(\mathbf{x}_i^{\neg n*} + \Delta\mathbf{x}_i^{\neg n*}, x_{in}^* + \Delta x_{in}^* - \Delta s_{in} + \frac{\Delta s_n(a^*)}{I}\right) > u_i(\mathbf{x}_i^{\neg n*} + \Delta\mathbf{x}_i^{\neg n*}, x_{in}^* + \Delta x_{in}^* - \Delta s_{in}) \geq u_i(\mathbf{x}_i^*).$$

The allocation underconsideration is thus P-improving as compared to a^* *i.e.* a^* is not P-efficient.
(2) $l_n(a^*) = 0$, $\forall n \in \mathcal{N} \Rightarrow a^*$ P-efficient. This restates: a^* not P-efficient $\Rightarrow \exists n \in \mathcal{N}$ s.t. $l_n(a^*) > 0$. Since a^* is not P-efficient, there exists $a = \left((\mathbf{x}_i)_{i\in\mathcal{I}}, (\mathbf{x}_j)_{j\in\mathcal{J}}\right)$ which is P-improving as compared to a^*, *i.e.* such that: $\forall i \in \mathcal{I}$, $u_i(\mathbf{x}_i) \geq u_i(\mathbf{x}_i^*)$ with at least one strict inequality. Let $\bar{\imath}$ index an individual for whom $u_{\bar{\imath}}(\mathbf{x}_{\bar{\imath}}) > u_{\bar{\imath}}(\mathbf{x}_{\bar{\imath}}^*)$ and consider the reallocation $\Delta a = \left((\Delta\mathbf{x}_i)_{i\in\mathcal{I}}, (\Delta\mathbf{x}_j)_{j\in\mathcal{J}}\right)$ defined by: $\Delta\mathbf{x}_i = \mathbf{x}_i - \mathbf{x}_i^*$ for all $i \in \mathcal{I} - \{\bar{\imath}\}$ and $\Delta\mathbf{x}_{\bar{\imath}} = (\mathbf{x}_{\bar{\imath}}^{\neg n} - \mathbf{x}_{\bar{\imath}}^{\neg n*}, x_{\bar{\imath}n} - x_{\bar{\imath}n}^* - \Delta s_{\bar{\imath}n})$ where $\Delta s_{\bar{\imath}n} > 0$ is defined by $u_i(\mathbf{x}_{\bar{\imath}}^{\neg n}, x_{\bar{\imath}n} - \Delta s_{\bar{\imath}n}) = u_i(\mathbf{x}_{\bar{\imath}}^*)$. Such a reallocation is feasible and releases the surplus $\Delta s_n = \Delta s_{\bar{\imath}n} > 0$. By definition, $l_n(a^*) \geq \Delta s_n$ and thus $l_n(a^*) > 0$. ■

B The TGS with "money"

Proof 16. For any reallocation $\Delta a = \left((\Delta\mathbf{x}_i, \Delta y_i)_{i\in\mathcal{I}}, (\Delta\mathbf{x}_j)_{j\in\mathcal{J}}\right)$, define $\bar{\Delta} a = \left((\bar{\Delta}\mathbf{x}_i, \bar{\Delta} y_i)_{i\in\mathcal{I}}, (\bar{\Delta}\mathbf{x}_j)_{j\in\mathcal{J}}\right)$ by $\left(\bar{\Delta}\mathbf{x}_i, \bar{\Delta}\mathbf{x}_j\right) = (\Delta\mathbf{x}_i, \Delta\mathbf{x}_j)$ for all $(i,j) \in \mathcal{I} \times \mathcal{J}$ but $\bar{\Delta} y_i \neq \Delta y_i$ for some $i \in \mathcal{I}$. For all $i \in \mathcal{I}$, by definition of individual surplus:

$$u_i(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i - \Delta s_i) = u_i\left(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \bar{\Delta} y_i - \bar{\Delta} s_i\right) = u_i(\mathbf{x}_i, y_i),$$

that is $\Delta y_i - \Delta s_i = \bar{\Delta} y_i - \bar{\Delta} s_i$. As a consequence

$$\bar{\Delta} s = \sum_{i\in\mathcal{I}} \bar{\Delta} s_i - \sum_{i\in\mathcal{I}} \bar{\Delta} y_i = \sum_{i\in\mathcal{I}} \Delta s_i - \sum_{i\in\mathcal{I}} \Delta y_i = \Delta s.$$

■

Proof 20. Since, for all $i \in \mathcal{I}$, $u_i(\mathbf{x}_i, y_i)$ is strictly increasing in y_i, $\Delta\mathfrak{s}_i(u_i; \Delta a)$ is strictly decreasing in u_i. So the conclusion. ■

Proof 18. (1) $l_n(a) = 0$ for all $n \in \mathcal{N} \Rightarrow l(a) = 0$. The statement is equivalent to: $l(a) > 0 \Rightarrow \exists n \in \mathcal{N}, l_n(a) > 0$. $l(a) > 0 : \exists \Delta a = \left((\Delta\mathbf{x}_i, \Delta y_i)_{i\in\mathcal{I}}, (\Delta\mathbf{x}_j)_{j\in\mathcal{J}}\right)$ feasible such that $\Delta s(a) > 0$ where $\Delta s(a) = \sum_{i\in\mathcal{I}} (\Delta s_i - \Delta y_i)$ and Δs_i such that $u_i(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i - \Delta s_i) = u_i(\mathbf{x}_i, y_i)$. $\Delta s(a) > 0 \Rightarrow \exists \bar{\imath} \in \mathcal{I}, \Delta s_{\bar{\imath}} - \Delta y_{\bar{\imath}} > 0$ or $u_{\bar{\imath}}(\mathbf{x}_{\bar{\imath}} + \Delta\mathbf{x}_{\bar{\imath}}, y_{\bar{\imath}}) > u_{\bar{\imath}}(\mathbf{x}_{\bar{\imath}}, y_{\bar{\imath}})$ which involves $\exists \bar{n} \in \mathcal{N}, \Delta x_{\bar{\imath}\bar{n}} > 0$. Let t be such that $u_{\bar{\imath}}(\mathbf{x}_{\bar{\imath}}^{\neg\bar{n}} + \Delta\mathbf{x}_{\bar{\imath}}^{\neg\bar{n}}, x_{\bar{\imath}\bar{n}}, y_{\bar{\imath}} + \Delta y_{\bar{\imath}} - \Delta s_{\bar{\imath}} + t) = u_{\bar{\imath}}(\mathbf{x}_{\bar{\imath}}, y_{\bar{\imath}})$: the assumptions made as regards "money" guarantee that t exists, furthermore, $t < \Delta s(a)$. Now, consider the reallocation $\bar{\Delta} a = \left((\bar{\Delta}\mathbf{x}_i, \bar{\Delta} y_i)_{i\in\mathcal{I}}, (\bar{\Delta}\mathbf{x}_j)_{j\in\mathcal{J}}\right)$ defined by: $\bar{\Delta}\mathbf{x}_j = \Delta\mathbf{x}_j$ for all $j \in \mathcal{J}$, $\bar{\Delta}\mathbf{x}_i = \Delta\mathbf{x}_i$ for all $i \in \mathcal{I}$, while for all $i \neq \bar{\imath}$, $\bar{\Delta} y_i = \Delta y_i - \Delta s_i$, and $\bar{\Delta} y_{\bar{\imath}} = \Delta y_{\bar{\imath}} - \Delta s_{\bar{\imath}} + t$. This reallocation is feasible:

$$\begin{aligned} \sum_{i\in\mathcal{I}} \bar{\Delta}\mathbf{x}_i + \sum_{j\in\mathcal{J}} \bar{\Delta}\mathbf{x}_j &= \sum_{i\in\mathcal{I}} \Delta\mathbf{x}_i + \sum_{j\in\mathcal{J}} \Delta\mathbf{x}_j \leq 0 \\ f_j\left(\mathbf{x}_j + \bar{\Delta}\mathbf{x}_j\right) &= f_j(\mathbf{x}_j + \Delta\mathbf{x}_j) \geq 0 \text{ for all } j \in \mathcal{J} \end{aligned}$$

and $\sum_{i\in\mathcal{I}} \bar{\Delta} y_i = \sum_{i\neq\bar{\imath}} (\Delta y_i - \Delta s_i) + \Delta y_{\bar{\imath}} - \Delta s_{\bar{\imath}} + t = -\Delta s\,(a) + t \leq 0$. For all $i \in \mathcal{I} - \{\bar{\imath}\}$ and $n \in \mathcal{N}$: $u_i\left(\mathbf{x}_i + \bar{\Delta}\mathbf{x}_i, y_i + \bar{\Delta} y_i\right) = u_i\left(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i - \Delta s_i\right) = u_i\left(\mathbf{x}_i, y_i\right)$,whereas

$$u_i\left(\mathbf{x}_{\bar{\imath}}^{\neg\bar{n}} + \bar{\Delta}\mathbf{x}_{\bar{\imath}}^{\neg\bar{n}}, x_{\bar{\imath}\bar{n}} + \bar{\Delta} x_{\bar{\imath}\bar{n}}, y_{\bar{\imath}} + \bar{\Delta} y_{\bar{\imath}}\right) = u_i\left(\mathbf{x}_{\bar{\imath}}^{\neg\bar{n}} + \bar{\Delta}\mathbf{x}_{\bar{\imath}}^{\neg\bar{n}}, x_{\bar{\imath}\bar{n}} + \Delta x_{\bar{\imath}\bar{n}}, y_{\bar{\imath}} + \Delta y_{\bar{\imath}} - \Delta s_{\bar{\imath}} + t\right),$$

with $u_i\left(\mathbf{x}_{\bar{\imath}}^{\neg\bar{n}} + \bar{\Delta}\mathbf{x}_{\bar{\imath}}^{\neg\bar{n}}, x_{\bar{\imath}\bar{n}} + \Delta x_{\bar{\imath}\bar{n}}, y_{\bar{\imath}} + \Delta y_{\bar{\imath}} - \Delta s_{\bar{\imath}} + t\right) > u_i\left(\mathbf{x}_{\bar{\imath}}^{\neg\bar{n}} + \bar{\Delta}\mathbf{x}_{\bar{\imath}}^{\neg\bar{n}}, x_{\bar{\imath}\bar{n}}, y_{\bar{\imath}} + \Delta y_{\bar{\imath}} - \Delta s_{\bar{\imath}} + t\right) = u_i\left(\mathbf{x}_{\bar{\imath}}, y_{\bar{\imath}}\right)$. It follows that for all $i \neq \bar{\imath}$, $\bar{\Delta} s_{i\bar{n}} = 0$, whereas $\bar{\Delta} s_{\bar{\imath}\bar{n}} = \Delta x_{\bar{\imath}\bar{n}} > 0$, and:

$$\bar{\Delta} s_{\bar{n}} = \sum_{i\in\mathcal{I}} \left(\bar{\Delta} s_{i\bar{n}} - \bar{\Delta} x_{i\bar{n}}\right) = \sum_{i\in\mathcal{I}} \bar{\Delta} s_{i\bar{n}} - \sum_{i\in\mathcal{I}} \bar{\Delta} x_{i\bar{n}} = \Delta x_{\bar{\imath}\bar{n}} - \sum_{i\in\mathcal{I}} \Delta x_{i\bar{n}} \geq \Delta x_{\bar{\imath}\bar{n}}.$$

Since $l_{\bar{n}}(a) \geq \bar{\Delta} s_{\bar{n}} > 0 : l(a) > 0 \Rightarrow \exists n \in \mathcal{N}, l_n(a) > 0$.
(2) $l(a) = 0 \Rightarrow l_n(a) = 0$ for all $n \in \mathcal{N}$. The statement is equivalent to: $\exists \bar{n} \in \mathcal{N}, l_{\bar{n}}(a) > 0 \Rightarrow l(a) > 0$. $\exists \bar{\Delta} a = \left(\left(\bar{\Delta}\mathbf{x}_i, \bar{\Delta} y_i\right)_{i\in\mathcal{I}}, \left(\bar{\Delta}\mathbf{x}_j\right)_{j\in\mathcal{J}}\right)$ feasible and releasing a strictly positive surplus as measured in good $\bar{n}$, let $\bar{\Delta} s_{\bar{n}}(a)$ denote corresponding surplus:

$$\bar{\Delta} s_{\bar{n}}(a) = \sum_{i\in\mathcal{I}} \left(\bar{\Delta} s_{i\bar{n}} - \bar{\Delta} x_{i\bar{n}}\right),$$

where, for all $i \in \mathcal{I} : \bar{\Delta} s_{i\bar{n}} = \max\left\{\Delta\sigma_{i\bar{n}} \mid u_i\left(\mathbf{x}_i^{\neg\bar{n}} + \bar{\Delta}\mathbf{x}_i^{\neg\bar{n}}, x_{i\bar{n}} + \bar{\Delta} x_{i\bar{n}} - \Delta\sigma_{i\bar{n}}, y_i + \bar{\Delta} y_i\right) = u_i\left(\mathbf{x}_i, y_i\right)\right\}$. From a, let's consider the reallocation $\Delta a = \left(\left(\Delta\mathbf{x}_i, \Delta y_i\right)_{i\in\mathcal{I}}, \left(\Delta\mathbf{x}_j\right)_{j\in\mathcal{J}}\right)$ defined, for all $i \in \mathcal{I}$, by $\Delta x_{in} = \bar{\Delta} x_{in}$ for all $n \neq \bar{n}$, $\Delta x_{in} =$ ■

$a^* \in \mathcal{A}$ Pareto-efficient $\Leftrightarrow \Delta s\,(a^*) \leq 0$ for all feasible reallocation $\Delta a \Leftrightarrow l\,(a^*) = 0$.

Proof 19. Two stages.
(1) $a \in \mathcal{A}$ P-efficient $\Rightarrow \Delta s\,(a) \leq 0$ for all feasible reallocation Δa. This restates: $a \in \mathcal{A}$ P-efficient $\Rightarrow \nexists \Delta a$ feasible such that $\Delta s\,(a) > 0$. It is shown that: $\exists \Delta a$ feasible such that $\Delta s\,(a) > 0 \Rightarrow a$ *not* P-efficient. Let $\Delta a = \left(\left(\Delta\mathbf{x}_i, \Delta y_i\right)_{i\in\mathcal{I}}, \left(\Delta\mathbf{x}_j\right)_{j\in\mathcal{J}}\right)$ be feasible and such that $\Delta s\,(a) > 0$. To Δa can be associated the allocation $\tilde{a} = \left(\left(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i - \Delta s_i\right)_{i\in\mathcal{I}}, \left(\mathbf{x}_j + \Delta\mathbf{x}_j\right)_{j\in\mathcal{J}}\right)$ isohedonic to a, that is such that, for all $i \in \mathcal{I} : u_i\left(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i - \Delta s_i\right) = u_i\left(\mathbf{x}_i, y_i\right)$. If $\Delta s\,(a) > 0$, some allocations exist which are feasible and P-improving as compared to a. Take for instance

$$\left(\left(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i - \Delta s_i + \frac{\Delta s\,(a)}{I}\right)_{i\in\mathcal{I}}, \left(\mathbf{x}_j + \Delta\mathbf{x}_j\right)_{j\in\mathcal{J}}\right).$$

Since Δa is feasible, this allocation is feasible as well, in particular

$$\sum_{i\in\mathcal{I}} \left(\Delta y_i - \Delta s_i + \frac{\Delta s\,(a)}{I}\right) = \sum_{i\in\mathcal{I}} \Delta y_i - \sum_{i\in\mathcal{I}} \Delta s_i + I\frac{\Delta s\,(a)}{I} = \sum_{i\in\mathcal{I}} \Delta y_i \leq 0.$$

Furthermore, since $u_i(\mathbf{x}_i, y_i)$ is strictly increasing in y_i, for all $i \in \mathcal{I}$:

$$u_i\left(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i - \Delta s_i + \frac{\Delta s(a)}{I}\right) > u_i(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i - \Delta s_i) = u_i(\mathbf{x}_i, y_i)$$

that is, the allocation considered is P-improving as compared to a. Thus, a is not P-efficient.

(2) $\Delta s(a) \leq 0$ for all feasible reallocation $\Delta a \Rightarrow a$ P-efficient. Let's consider the equivalent statement that a P-inefficient $\Rightarrow \exists \Delta a$ feasible such that $\Delta s(a) > 0$. Let $\hat{a} = \left((\hat{\mathbf{x}}_i, \hat{y}_i)_{i\in\mathcal{I}}, (\hat{\mathbf{x}}_j)_{j\in\mathcal{J}}\right) \in \mathcal{A}$ be feasible and P-improving as compared to some allocation $a : \forall i \in \mathcal{I}, u_i(\hat{\mathbf{x}}_i, \hat{y}_i) \geq u_i(\mathbf{x}_i, y_i)$ with at least one strict inequality. From a, let's consider the reallocation $\Delta a = \left((\Delta\mathbf{x}_i, \Delta y_i)_{i\in\mathcal{I}}, (\Delta\mathbf{x}_j)_{j\in\mathcal{J}}\right)$ defined by $u_i(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i) = u_i(\hat{\mathbf{x}}_i, \hat{y}_i)$ for all $i \in \mathcal{I}$. Since a and $\hat{a}$ are both feasible, Δa is feasible as well. Furthermore, for all $i \in \mathcal{I} : u_i(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i) \geq u_i(\mathbf{x}_i, y_i)$ with at least one strict inequality. Since $u_i(\mathbf{x}_i, y_i)$ is strictly increasing in y_i (which varies continuously): $\forall i \in \mathcal{I}, \exists \Delta s_i \geq 0$ such that $u_i(\mathbf{x}_i + \Delta\mathbf{x}_i, y_i + \Delta y_i - \Delta s_i) = u_i(\mathbf{x}_i, y_i)$ with $\Delta s_i > 0$ for at least one $i \in \mathcal{I}$. It follows that the surplus $\Delta s(a)$ associated to Δa from the allocation $a \in \mathcal{A}$ satisfies $\Delta s(a) = \sum_{i\in\mathcal{I}} \Delta s_i > 0$. ■

C The marginal TGS

C.1 The TGS in terms of marginal valuations

Proof 23. For any $i \in \mathcal{I}$ and any infinitesimal individual change $(d\mathbf{x}_i, dy_i)$, the definition of individual surplus ds_i involves

$$\sum_{n\in\mathcal{N}} u'_{in} dx_{in} + u'_{iy} \cdot (dy_i - ds_i) = 0,$$

which, since $u'_{iy} > 0$, can be rewritten

$$ds_i = \frac{du_i}{u'_{iy}} = \sum_{n\in\mathcal{N}} \frac{u'_{in}}{u'_{iy}} dx_{in} + dy_i,$$

where du_i is the change in utility level induced by $(d\mathbf{x}_i, dy_i)$. In terms of marginal subjective valuations:

$$ds_i = \sum_{n\in\mathcal{N}} v'_{in} dx_{in} + dy_i.$$

For any infinitesimal reallocation $da = \left((d\mathbf{x}_i, dy_i)_{i\in\mathcal{I}}, (d\mathbf{x}_j)_{j\in\mathcal{I}}\right)$, the definition of collective surplus leads to:

$$\begin{aligned} ds &= \sum_{i\in\mathcal{I}} ds_i - \sum_{i\in\mathcal{I}} dy_i, \\ ds &= \sum_{i\in\mathcal{I}} \sum_{n\in\mathcal{N}} v'_{in} dx_{in}. \end{aligned}$$

■

Proof 25. By definition, for all $i \in \mathcal{I}$:

$$\begin{aligned} du_i &= \sum_{n \neq \underline{n}} u'_{in} dx_{in} + u'_{i\underline{n}} \cdot \left(dx_{i\underline{n}} - d\underline{s}_i \right) + u'_{iy} dy_i = 0 \\ &\Rightarrow \quad d\underline{s}_i - \sum_{n \neq \underline{n}} \underline{v}'_{in} dx_{in} - \underline{v}'_{iy} dy_i = dx_{i\underline{n}}, \end{aligned}$$

summing with respect to i :

$$\sum_{i \in \mathcal{I}} d\underline{s}_i - \sum_{i \in \mathcal{I}} \sum_{n \neq \underline{n}} \underline{v}'_{in} dx_{in} - \sum_{i \in \mathcal{I}} \underline{v}'_{iy} dy_i = \sum_{i \in \mathcal{I}} dx_{i\underline{n}}.$$

For all $j \in \mathcal{J}$:

$$\begin{aligned} dl_j &= \sum_{n \neq \underline{n}} f'_{jn} dx_{jn} + f'_{j\underline{n}} dx_{j\underline{n}} \\ &\Rightarrow \quad \frac{dl_j}{f'_{j\underline{n}}} = \sum_{n \neq \underline{n}} \frac{f'_{jn}}{f'_{j\underline{n}}} dx_{jn} + dx_{j\underline{n}}, \end{aligned}$$

denoting $d\underline{l}_j = \frac{dl_j}{f'_{j\underline{n}}}$, the transformation loss associated to j, as measured in labor, this can be rewritten

$$d\underline{l}_j - \sum_{n \neq \underline{n}} \underline{v}'_{jn} dx_{jn} = dx_{j\underline{n}},$$

and summing with respect to j leads to:

$$\sum_{j \in \mathcal{J}} d\underline{l}_j - \sum_{j \in \mathcal{J}} \sum_{n \neq \underline{n}} \underline{v}'_{jn} dx_{jn} = \sum_{j \in \mathcal{J}} dx_{j\underline{n}}.$$

Use-supply balance as regards labor induces:

$$\sum_{i \in \mathcal{I}} x_{i\underline{n}} + \sum_{j \in \mathcal{J}} x_{j\underline{n}} = 0 \Rightarrow \sum_{i \in \mathcal{I}} dx_{i\underline{n}} + \sum_{j \in \mathcal{J}} dx_{j\underline{n}} = 0,$$

and thus, with obvious writings

$$\sum_{i \in \mathcal{I}} d\underline{s}_i - \sum_{i \in \mathcal{I}} \sum_{n \neq \underline{n}} \underline{v}'_{in} dx_{in} - \sum_{i \in \mathcal{I}} \underline{v}'_{iy} dy_i + \sum_{j \in \mathcal{J}} d\underline{l}_j - \sum_{j \in \mathcal{J}} \sum_{n \neq \underline{n}} \underline{v}'_{jn} dx_{jn} = 0,$$
$$d\underline{s} = \sum_{i \in \mathcal{I}} \sum_{n \neq \underline{n}} \underline{v}'_{in} dx_{in} + \sum_{j \in \mathcal{J}} \sum_{n \neq \underline{n}} \underline{v}'_{jn} dx_{jn} + \sum_{i \in \mathcal{I}} \underline{v}'_{iy} dy_i - d\underline{l}_{\mathcal{J}}.$$

For all $n \neq \underline{n}$:

$$\begin{aligned} dx_{in} &= \sum_{k \in \mathcal{K}} d_k x_{in} = \sum_{\hat{\imath} \neq i} d_{\hat{\imath}} x_{in} + \sum_{j \in \mathcal{J}} d_j x_{in}, \; dy_i = \sum_{\hat{\imath} \neq i} d_{\hat{\imath}} y_i, \\ dx_{jn} &= \sum_{k \in \mathcal{K}} d_k x_{jn} = \sum_{\hat{\jmath} \neq j} d_{\hat{\jmath}} x_{jn} + \sum_{i \in \mathcal{I}} d_i x_{jn}, \end{aligned}$$

so that

$$
\begin{aligned}
d\underline{s} &= \sum_{n\neq\underline{n}}\left(\begin{array}{c}\sum_{i\in\mathcal{I}}\underline{v}'_{in}\cdot\left(\sum_{\hat{\imath}\neq i}d_{\hat{\imath}}x_{in}+\sum_{j\in\mathcal{J}}d_jx_{in}\right)\\ +\sum_{j\in\mathcal{J}}\underline{v}'_{jn}\cdot\left(\sum_{\hat{\jmath}\neq j}d_{\hat{\jmath}}x_{jn}+\sum_{i\in\mathcal{I}}d_ix_{jn}\right)\end{array}\right)+\sum_{i\in\mathcal{I}}\underline{v}'_{iy}\cdot\left(\sum_{\hat{\imath}\neq i}d_{\hat{\imath}}y_i\right)-d\underline{l}_{\mathcal{J}},\\
d\underline{s} &= \sum_{n\neq\underline{n}}\left(\begin{array}{c}\sum_{i\in\mathcal{I}}\sum_{\hat{\imath}\neq i}\underline{v}'_{in}d_{\hat{\imath}}x_{in}+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\underline{v}'_{in}d_jx_{in}\\ +\sum_{j\in\mathcal{J}}\sum_{\hat{\jmath}\neq j}\underline{v}'_{jn}d_{\hat{\jmath}}x_{jn}+\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\underline{v}'_{jn}d_ix_{jn}\end{array}\right)+\sum_{i\in\mathcal{I}}\sum_{\hat{\imath}\neq i}\underline{v}'_{iy}d_{\hat{\imath}}y_i-d\underline{l}_{\mathcal{J}},
\end{aligned}
$$

that is, with $d_{\hat{k}}x_{kn}=-d_kx_{\hat{k}n}$:

$$
\begin{aligned}
\sum_{i\in\mathcal{I}}\sum_{\hat{\imath}\neq i}\underline{v}'_{in}d_{\hat{\imath}}x_{in} &= \sum_{i\in\mathcal{I}}\sum_{\hat{\imath}>i}\left(\underline{v}'_{in}-\underline{v}'_{\hat{\imath}n}\right)d_{\hat{\imath}}x_{in}\\
\sum_{j\in\mathcal{J}}\sum_{\hat{\jmath}\neq j}\underline{v}'_{jn}d_{\hat{\jmath}}x_{jn} &= \sum_{j\in\mathcal{J}}\sum_{\hat{\jmath}>j}\left(\underline{v}'_{jn}-\underline{v}'_{\hat{\jmath}n}\right)d_{\hat{\jmath}}x_{jn}\\
\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\underline{v}'_{in}d_jx_{in}+\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\underline{v}'_{jn}d_ix_{jn} &= \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\left(\underline{v}'_{in}-\underline{v}'_{jn}\right)d_jx_{in}
\end{aligned}
$$

and

$$
d\underline{s}=\sum_{n\neq\underline{n}}\sum_{k\in\mathcal{K}}\sum_{\hat{k}>k}\left(\underline{v}'_{kn}-\underline{v}'_{\hat{k}n}\right)d_{\hat{k}}x_{kn}+\sum_{i\in\mathcal{I}}\sum_{\hat{\imath}>i}\left(\underline{v}'_{iy}-\underline{v}'_{\hat{\imath}y}\right)d_{\hat{\imath}}y_i-d\underline{l}_{\mathcal{J}}.
$$

■

C.2 Infinitesimal variations of surplus

Proof 26. The definition $v'_{in}\equiv\frac{u'_{in}}{u'_{iy}}\Rightarrow dv'_{in}=\frac{du'_{in}u'_{iy}-u'_{in}du'_{iy}}{\left(u'_{iy}\right)^2}=\frac{du'_{in}}{u'_{iy}}-v'_{in}\frac{du'_{iy}}{u'_{iy}}$ or $\frac{du'_{in}}{u'_{iy}}=dv'_{in}+v'_{in}\frac{du'_{iy}}{u'_{iy}}$. Hence:

$$
\begin{aligned}
d^2u_i &= \sum_{n\in\mathcal{N}}du'_{in}dx_{ni}+\sum_{n\in\mathcal{N}}u'_{in}d^2x_n+du'_{iy}\cdot(dy_i-ds_i)+u'_{iy}\cdot\left(d^2y_i-d^2s_i\right),\\
\frac{d^2u_i}{u'_{iy}} &= \sum_{n\in\mathcal{N}}\frac{du'_{in}}{u'_{iy}}dx_{in}+\frac{du'_{iy}}{u'_{iy}}\cdot(dy_i-ds_i)+\sum_{n\in\mathcal{N}}v'_{in}d^2x_{in}+\left(d^2y_i-d^2s_i\right),\\
\frac{d^2u_i}{u'_{iy}} &= \sum_{n\in\mathcal{N}}\left(dv'_{in}+v'_{in}\frac{du'_{iy}}{u'_{iy}}\right)dx_{in}+\frac{du'_{iy}}{u'_{iy}}\cdot(dy_i-ds_i)+\sum_{n\in\mathcal{N}}v'_{in}d^2x_{in}+\left(d^2y_i-d^2s_i\right),\\
\frac{d^2u_i}{u'_{iy}} &= \sum_{n\in\mathcal{N}}dv'_{in}dx_{in}+\left(\sum_{n\in\mathcal{N}}v'_{in}dx_{in}+dy_i-ds_i\right)\frac{du'_{iy}}{u'_{iy}}+\sum_{n\in\mathcal{N}}v'_{in}d^2x_{in}+d^2y_i-d^2s_i,
\end{aligned}
$$

where $\frac{du'_{iy}}{u'_{iy}} = \sum_{n\in\mathcal{N}} \frac{u''_{iny}}{u'_{iy}} dx_{in} + \frac{u''_{iy2}}{u'_{iy}} (dy_i - ds_i)$ and $v''_{iny} = \frac{u''_{iny}}{u'_{iy}} - v'_{in} \frac{u''_{iy2}}{u'_{iy}} \Rightarrow \frac{du'_{iy}}{u'_{iy}} =$
$\sum_{n\in\mathcal{N}} \left(v''_{iny} + v'_{in} \frac{u''_{iy2}}{u'_{iy}} \right) dx_{in} + \frac{u''_{iy2}}{u'_{iy}} (dy_i - ds_i) = \sum_{n\in\mathcal{N}} v''_{iny} dx_{in} + \left(\sum_{n\in\mathcal{N}} v'_{in} dx_{in} + dy_i - ds_i \right) \frac{u''_{iy2}}{u'_{iy}}$
so that

$$\begin{aligned}
\frac{d^2 u_i}{u'_{iy}} &= \sum_{n\in\mathcal{N}} \left(\sum_{\bar{n}\in\mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} + v''_{iny} \cdot (dy_i - ds_i) \right) dx_{in} \\
&\quad + \left(\sum_{n\in\mathcal{N}} v'_{in} dx_{in} + dy_i - ds_i \right) \left(\sum_{n\in\mathcal{N}} v''_{iny} dx_{in} + \left(\sum_{n\in\mathcal{N}} v'_{in} dx_{in} + dy_i - ds_i \right) \frac{u''_{iy^2}}{u'_{iy}} \right) \\
&\quad + \sum_{n\in\mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i, \\
\frac{d^2 u_i}{u'_{iy}} &= \sum_{n\in\mathcal{N}} \sum_{\bar{n}\in\mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} dx_{in} + \sum_{n\in\mathcal{N}} v''_{iny} \cdot (dy_i - ds_i) \, dx_{in} \\
&\quad + \left(\sum_{n\in\mathcal{N}} v'_{in} dx_{in} + dy_i - ds_i \right) \sum_{n\in\mathcal{N}} v''_{iny} dx_{in} + \left(\sum_{n\in\mathcal{N}} v'_{in} dx_{in} + dy_i - ds_i \right)^2 \frac{u''_{iy^2}}{u'_{iy}} \\
&\quad + \sum_{n\in\mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i, \\
\frac{d^2 u_i}{u'_{iy}} &= \sum_{n\in\mathcal{N}} \sum_{\bar{n}\in\mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} dx_{in} + \sum_{n\in\mathcal{N}} v'_{in} dx_{in} \sum_{n\in\mathcal{N}} v''_{iny} dx_{in} + 2 (dy_i - ds_i) \sum_{n\in\mathcal{N}} v''_{iny} dx_{in} \\
&\quad + \left(\sum_{n\in\mathcal{N}} v'_{in} dx_{in} + dy_i - ds_i \right)^2 \frac{u''_{iy^2}}{u'_{iy}} \\
&\quad + \sum_{n\in\mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i,
\end{aligned}$$

with $ds_i = \sum_{n\in\mathcal{N}} v'_{in} dx_{in} + dy_i$ so that

$$\begin{aligned}
\frac{d^2u_i}{u'_{iy}} &= \sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} dx_{in} + \sum_{n\in\mathcal{N}} v'_{in} dx_{in} \sum_{n\in\mathcal{N}} v''_{iny} dx_{in} - 2\sum_{n\in\mathcal{N}} v'_{in} dx_{in} \sum_{n\in\mathcal{N}} v''_{iny} dx_{in} \\
&\quad + \sum_{n\in\mathcal{N}} v'_{in} d^2x_{in} + d^2y_i - d^2s_i, \\
\frac{d^2u_i}{u'_{iy}} &= \sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} dx_{in} - \sum_{n\in\mathcal{N}} v'_{in} dx_{in} \sum_{n\in\mathcal{N}} v''_{iny} dx_{in} \\
&\quad + \sum_{n\in\mathcal{N}} v'_{in} d^2x_{in} + d^2y_i - d^2s_i, \\
\frac{d^2u_i}{u'_{iy}} &= \sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} dx_{in} - \sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}} v'_{in} v''_{i\bar{n}y} dx_{i\bar{n}} dx_{in} \\
&\quad + \sum_{n\in\mathcal{N}} v'_{in} d^2x_{in} + d^2y_i - d^2s_i, \\
\frac{d^2u_i}{u'_{iy}} &= \sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}} \left(v''_{in\bar{n}} - v'_{in} v''_{i\bar{n}y}\right) dx_{i\bar{n}} dx_{in} + \sum_{n\in\mathcal{N}} v'_{in} d^2x_{in} + d^2y_i - d^2s_i,
\end{aligned}$$

Conclusion $\frac{d^2u_i}{u'_{iy}} = 0$ involves

$$d^2s_i = \sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}} \left(v''_{in\bar{n}} - v'_{i\bar{n}} v''_{iny}\right) dx_{i\bar{n}} dx_{in} + \sum_{n\in\mathcal{N}} v'_{in} d^2x_{in} + d^2y_i.$$

■

C.3 Pareto-efficiency and walrasian general equilibrium

Proof. $28a \in \mathcal{A}$ Pareto-efficient $\Leftrightarrow \Delta s(a) \leq 0$ for all feasible reallocations. This means that the total (unobserved) surplus[29] $s(a)$ is maximal in a. Provided that functions $(u_i(.))_{i\in\mathcal{I}}$ are continuous and derivable in all directions, this occures under the necessary and sufficient condition that $ds(a) = 0$ and $d^2s(a) \leq 0$. ■

Proof 29. Let's consider a feasible infinitesimal reallocation $\left((d\mathbf{x}_i, dy_i)_{i\in\mathcal{I}}, (d\mathbf{x}_j)_{j\in\mathcal{J}}\right)$ involving no loss increase, that is such that

$$\begin{aligned}
\sum_{i\in\mathcal{I}} dy_i &= 0 \text{ and } \sum_{i\in\mathcal{I}} dx_{in} + \sum_{j\in\mathcal{J}} dx_{jn} = 0 \text{ for all } n \in \mathcal{N} \\
\sum_{n\in\mathcal{N}} f'_{jn} dx_{jn} &= 0 \text{ for all } j \in \mathcal{J}
\end{aligned}$$

(1) and (2) $\Rightarrow ds = 0$. Consider $\hat{n} \in \mathcal{N}$ be such that $f'_{j\hat{n}} > 0$ for any $j \in \mathcal{J}$: $\sum_{n\in\mathcal{N}} f'_{jn} dx_{jn} = 0 \Leftrightarrow \sum_{n\in\mathcal{N}} \frac{f'_{jn}}{f'_{j\hat{n}}} dx_{jn} = 0$ by (2) : $\sum_{n\in\mathcal{N}} \frac{v'_{in}}{v'_{i\hat{n}}} dx_{jn} = \sum_{n\in\mathcal{N}} \frac{f'_{jn}}{f'_{j\hat{n}}} dx_{jn} = 0$

[29] Defined by comparison to a state where each agent operates in autarky with his initial endowment.

and $\sum_{j\in\mathcal{J}}\sum_{n\in\mathcal{N}}\frac{v'_{in}}{v'_{i\hat{n}}}dx_{jn}=0$. By (1) one can define $p_n=v'_{in}$ and $p_{\hat{n}}=v'_{i\hat{n}}$. This entails $\frac{v'_{in}}{v'_{i\hat{n}}}=\frac{p_n}{p_{\hat{n}}}$ so that

$$\sum_{j\in\mathcal{J}}\sum_{n\in\mathcal{N}}\frac{v'_{in}}{v'_{i\hat{n}}}dx_{jn}=\sum_{j\in\mathcal{J}}\sum_{n\in\mathcal{N}}\frac{p_n}{p_{\hat{n}}}dx_{jn}=\sum_{n\in\mathcal{N}}\frac{p_n}{p_{\hat{n}}}\sum_{j\in\mathcal{J}}dx_{jn}=0$$

but $\sum_{j\in\mathcal{J}}dx_{jn}=-\sum_{i\in\mathcal{J}}dx_{in}$ (feasibility) so that

$$\sum_{n\in\mathcal{N}}\frac{p_n}{p_{\hat{n}}}\sum_{j\in\mathcal{J}}dx_{jn}=-\sum_{n\in\mathcal{N}}\frac{p_n}{p_{\hat{n}}}\sum_{i\in\mathcal{I}}dx_{in}=0\Rightarrow\frac{1}{p_{\hat{n}}}\sum_{i\in\mathcal{I}}\sum_{n\in\mathcal{N}}p_n dx_{in}=0$$

and hence, since $p_{\hat{n}}>0:\sum_{i\in\mathcal{I}}\sum_{n\in\mathcal{N}}v'_{in}dx_{in}=ds=0$.

If $ds=0$ then (1). This restates: if $ds=0$ then $\nexists\,(\check{\imath},\hat{\imath})\in\mathcal{I}^2,\hat{\imath}\neq\check{\imath}$, and $\bar{n}\in\mathcal{N}$ such that $v'_{\hat{\imath}\bar{n}}\neq v'_{\check{\imath}\bar{n}}$. It is shown that if $\exists\,(\check{\imath},\hat{\imath})\in\mathcal{I}^2,\hat{\imath}\neq\check{\imath}$, and $\bar{n}\in\mathcal{N}$ such that $v'_{\hat{\imath}\bar{n}}\neq v'_{\check{\imath}\bar{n}}$ then $ds\neq0$. With no loss in generality, let's consider $(\check{\imath},\hat{\imath})\in\mathcal{I}^2,\hat{\imath}\neq\check{\imath}$, and $\bar{n}\in\mathcal{N}$ such that $v'_{\hat{\imath}\bar{n}}>v'_{\check{\imath}\bar{n}}$, as well as the reallocation $da=\left((d\mathbf{x}_i,dy_i)_{i\in\mathcal{I}},(d\mathbf{x}_j)_{j\in\mathcal{J}}\right)$ defined by:

$$\begin{aligned}dx_{in}&=dy_i=0\text{ for all }i\in\mathcal{I}-\{\hat{\imath},\check{\imath}\}\text{ and }n\in\mathcal{N}\\ d\mathbf{x}_j&=\mathbf{0}\text{ for all }j\in\mathcal{J}\end{aligned}$$

but $dx_{\hat{\imath}\bar{n}}=-dx_{\check{\imath}\bar{n}}>0$ and $dy_{\hat{\imath}}=-dy_{\check{\imath}}\,(=-v'_{\hat{\imath}\bar{n}}dx_{\hat{\imath}\bar{n}}<0)$. One can check that this reallocation is feasible:

$$\begin{aligned}\sum_{i\in\mathcal{I}}dx_{in}+\sum_{j\in\mathcal{J}}dx_{jn}&=0\text{ for all }n\in\mathcal{N}-\{\bar{n}\}\\ \sum_{i\in\mathcal{I}}dx_{i\bar{n}}+\sum_{j\in\mathcal{J}}dx_{j\bar{n}}&=dx_{\check{\imath}\bar{n}}+dx_{\hat{\imath}\bar{n}}=0\\ \sum_{i\in\mathcal{I}}dy_i&=dy_{\check{\imath}}+dy_{\hat{\imath}}=0\end{aligned}$$

and yet:

$$\begin{aligned}ds&=\sum_{i\in\mathcal{I}}\sum_{n\in\mathcal{N}}v'_{in}dx_{in}=v'_{\hat{\imath}\bar{n}}dx_{\hat{\imath}\bar{n}}+v'_{\check{\imath}\bar{n}}dx_{\check{\imath}\bar{n}}\\ ds&=v'_{\hat{\imath}\bar{n}}dx_{\hat{\imath}\bar{n}}-v'_{\check{\imath}\bar{n}}dx_{\hat{\imath}\bar{n}}=\underbrace{(v'_{\hat{\imath}\bar{n}}-v'_{\check{\imath}\bar{n}})}_{>0}\underbrace{dx_{\hat{\imath}\bar{n}}}_{>0}>0\end{aligned}$$

If $ds=0$ then (2). This restates: if $ds=0$ then $\nexists\,(i,j)\in\mathcal{I}\times\mathcal{J}$ and $(n,\bar{n})\in\mathcal{N}$, $\bar{n}\neq n$ such that $\frac{f'_{jn}}{f'_{j\bar{n}}}\neq\frac{u'_{in}}{u'_{i\bar{n}}}$. It is shown that if $\exists\,(i,j)\in\mathcal{I}\times\mathcal{J}$ and $(n,\bar{n})\in\mathcal{N}$, $\bar{n}\neq n$ such that $\frac{f'_{jn}}{f'_{j\bar{n}}}\neq\frac{u'_{in}}{u'_{i\bar{n}}}$ then $ds\neq0$. With no loss in generality, let's consider $(\bar{\imath},\bar{\jmath}) \in \mathcal{I}\times\mathcal{J}$ and $(\check{n},\hat{n}) \in \mathcal{N}$, $\check{n} \neq \hat{n}$ such that $\frac{u'_{\bar{\imath}\hat{n}}}{u'_{\bar{\imath}\check{n}}} > \frac{f'_{\bar{\jmath}\hat{n}}}{f'_{\bar{\jmath}\check{n}}}$ as well as the reallocation $da = \left((d\mathbf{x}_i, dy_i)_{i\in\mathcal{I}}, (d\mathbf{x}_j)_{j\in\mathcal{J}}\right)$ defined by: $dx_{in} = dy_i = 0$ for all $i \in \mathcal{I} - \{\bar{\imath}\}$ and $n \in \mathcal{N}$, $dx_{jn} = 0$ for all $j \in \mathcal{J} - \{\bar{\jmath}\}$ and $n \in \mathcal{N}$, $dx_{\bar{\imath}n} = dx_{\bar{\jmath}n} = 0$ for all $n \in \mathcal{N} - \{\check{n},\hat{n}\}$, but $dx_{\bar{\imath}\hat{n}} = -dx_{\bar{\jmath}\hat{n}} > 0$ and $dx_{\bar{\imath}\check{n}} = -dx_{\bar{\jmath}\check{n}} = \frac{f'_{\bar{\jmath}\hat{n}}}{f'_{\bar{\jmath}\check{n}}} dx_{\bar{\jmath}\hat{n}} < 0$. One can check that this reallocation is feasible:

$$\begin{aligned}
\sum_{i\in\mathcal{I}} dx_{in} + \sum_{j\in\mathcal{J}} dx_{jn} &= 0 \text{ for all } n \in \mathcal{N} - \{\check{n},\hat{n}\} \\
\sum_{i\in\mathcal{I}} dx_{i\check{n}} + \sum_{j\in\mathcal{J}} dx_{j\check{n}} &= dx_{\bar{\imath}\check{n}} + dx_{\bar{\jmath}\check{n}} = 0 \\
\sum_{i\in\mathcal{I}} dx_{i\hat{n}} + \sum_{j\in\mathcal{J}} dx_{j\hat{n}} &= dx_{\bar{\imath}\hat{n}} + dx_{\bar{\jmath}\hat{n}} = 0 \\
\sum_{n\in\mathcal{N}} f'_{jn} dx_{jn} &= 0 \text{ for all } j \in \mathcal{J} - \{\bar{\jmath}\} \\
\sum_{i\in\mathcal{I}} dy_i &= dy_{\check{\imath}} + dy_{\hat{\imath}} = 0
\end{aligned}$$

and

$$\sum_{n\in\mathcal{N}} f'_{\bar{\jmath}n} dx_{\bar{\jmath}n} = f'_{\bar{\jmath}\check{n}} dx_{\bar{\jmath}\check{n}} + f'_{\bar{\jmath}\hat{n}} dx_{\bar{\jmath}\hat{n}} = f'_{\bar{\jmath}\check{n}} \cdot \left(-\frac{f'_{\bar{\jmath}\hat{n}}}{f'_{\bar{\jmath}\check{n}}} dx_{\bar{\jmath}\hat{n}}\right) + f'_{\bar{\jmath}\hat{n}} dx_{\bar{\jmath}\hat{n}} = 0$$

but still:

$$\begin{aligned}
ds &= \sum_{i\in\mathcal{I}}\sum_{n\in\mathcal{N}} \frac{u'_{in}}{u'_{iy}} dx_{in} = \frac{u'_{\bar{\imath}\check{n}}}{u'_{\bar{\imath}y}} dx_{\bar{\imath}\check{n}} + \frac{u'_{\bar{\imath}\hat{n}}}{u'_{\bar{\imath}y}} dx_{\bar{\imath}\hat{n}} \\
ds &= \frac{u'_{\bar{\imath}\check{n}}}{u'_{\bar{\imath}y}} \cdot \left(dx_{\bar{\imath}\check{n}} + \frac{u'_{\bar{\imath}\hat{n}}}{u'_{\bar{\imath}\check{n}}} dx_{\bar{\imath}\hat{n}}\right) = \frac{u'_{\bar{\imath}\check{n}}}{u'_{\bar{\imath}y}} \cdot \left(-\frac{f'_{\bar{\jmath}\hat{n}}}{f'_{\bar{\jmath}\check{n}}} dx_{\bar{\imath}\hat{n}} + \frac{u'_{\bar{\imath}\hat{n}}}{u'_{\bar{\imath}\check{n}}} dx_{\bar{\imath}\hat{n}}\right) \\
ds &= \underbrace{\frac{u'_{\bar{\imath}\check{n}}}{u'_{\bar{\imath}y}}}_{>0} \cdot \underbrace{\left(\frac{u'_{\bar{\imath}\hat{n}}}{u'_{\bar{\imath}\check{n}}} - \frac{f'_{\bar{\jmath}\hat{n}}}{f'_{\bar{\jmath}\check{n}}}\right)}_{>0} \underbrace{dx_{\bar{\imath}\hat{n}}}_{>0} > 0
\end{aligned}$$

■

Proof 30. $d^2 s$ is written

$$\begin{aligned}
d^2 s &= \sum_{i\in\mathcal{I}} \left(\sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}} \left(v''_{in\bar{n}} - v'_{in} v''_{i\bar{n}y}\right) dx_{i\bar{n}} dx_{in} + \sum_{n\in\mathcal{N}} v'_{in} d^2 x_{in}\right), \\
d^2 s &= \sum_{i\in\mathcal{I}}\sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}} \left(v''_{in\bar{n}} - v'_{in} v''_{i\bar{n}y}\right) dx_{i\bar{n}} dx_{in} + \sum_{n\in\mathcal{N}}\sum_{i\in\mathcal{I}} v'_{in} d^2 x_{in}.
\end{aligned}$$

It has been shown that $ds(a) = 0 \Rightarrow \exists (p_n)_{n\in\mathcal{N}}$ such that $\forall i \in \mathcal{I} : v'_{in} = p_n$. Consequently:

$$\sum_{n\in\mathcal{N}}\sum_{i\in\mathcal{I}} v'_{in} d^2 x_{in} = \sum_{n\in\mathcal{N}} p_n \sum_{i\in\mathcal{I}} d^2 x_{in}$$

Suppose the reallocation induces no waste: $\forall n \in \mathcal{N} : \sum_{i \in \mathcal{I}} dx_{in} + \sum_{j \in \mathcal{J}} dx_{jn} = 0 \Rightarrow \sum_{i \in \mathcal{I}} d^2 x_{in} + \sum_{j \in \mathcal{J}} d^2 x_{jn} = 0$ so that

$$\begin{aligned} \sum_{i \in \mathcal{I}} d^2 x_{in} &= -\sum_{j \in \mathcal{J}} d^2 x_{jn} \\ \sum_{n \in \mathcal{N}} p_n \sum_{i \in \mathcal{I}} d^2 x_{in} &= -\sum_{n \in \mathcal{N}} p_n \sum_{j \in \mathcal{J}} d^2 x_{jn} \end{aligned}$$

$ds(a) = 0 \Rightarrow$ the reallocation does not increase transformation losses in unit $j \in \mathcal{J}$:

$$\sum_{n \in \mathcal{N}} f'_{jn} dx_{jn} = 0$$

and

$$d\left(\sum_{n \in \mathcal{N}} f'_{jn} dx_{jn}\right) = \sum_{n \in \mathcal{N}} df'_{jn} dx_{jn} + \sum_{n \in \mathcal{N}} f'_{jn} d^2 x_{jn} = 0$$

Since this is the case for any transformation unit $j \in \mathcal{J}$:

$$\sum_{j \in \mathcal{J}} \sum_{n \in \mathcal{N}} df'_{jn} dx_{jn} + \sum_{j \in \mathcal{J}} \sum_{n \in \mathcal{N}} f'_{jn} d^2 x_{jn} = 0$$

Consider an input $\underline{n}$ required in the transformation process of any unit $j \in \mathcal{J}$ (e.g. labor) *i.e.* such that $f'_{j\underline{n}} > 0$ for all $j \in \mathcal{J}$. One gets

$$\sum_{j \in \mathcal{J}} \sum_{n \in \mathcal{N}} \frac{df'_{jn}}{f'_{j\underline{n}}} dx_{jn} + \sum_{j \in \mathcal{J}} \sum_{n \in \mathcal{N}} \frac{f'_{jn}}{f'_{j\underline{n}}} d^2 x_{jn} = 0$$

It has been shown that $ds(a) = 0 \Rightarrow \forall (i,j) \in \mathcal{I} \times \mathcal{J}$ and $n \in \mathcal{N} : \frac{f'_{jn}}{f'_{j\underline{n}}} = \frac{v'_{in}}{v'_{i\underline{n}}} = \frac{p_n}{p_{\underline{n}}}$. As a consequence, if $ds(a) = 0$:

$$\begin{aligned} \sum_{j \in \mathcal{J}} \sum_{n \in \mathcal{N}} \frac{df'_{jn}}{f'_{j\underline{n}}} dx_{jn} + \sum_{j \in \mathcal{J}} \sum_{n \in \mathcal{N}} \frac{p_n}{p_{\underline{n}}} d^2 x_{jn} &= 0 \\ \sum_{j \in \mathcal{J}} \sum_{n \in \mathcal{N}} \frac{df'_{jn}}{f'_{j\underline{n}}} dx_{jn} + \sum_{n \in \mathcal{N}} \frac{p_n}{p_{\underline{n}}} \sum_{j \in \mathcal{J}} d^2 x_{jn} &= 0 \\ \sum_{n \in \mathcal{N}} p_n \sum_{j \in \mathcal{J}} d^2 x_{jn} &= -p_{\underline{n}} \sum_{j \in \mathcal{J}} \sum_{n \in \mathcal{N}} \frac{df'_{jn}}{f'_{j\underline{n}}} dx_{jn} \end{aligned}$$

From which, it follows that

$$d^2s = \sum_{i\in\mathcal{I}}\sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}}\left(v''_{in\bar{n}} - v'_{in}v''_{i\bar{n}y}\right) dx_{i\bar{n}}dx_{in} - \sum_{n\in\mathcal{N}} p_n \sum_{j\in\mathcal{J}} d^2x_{jn}$$

$$d^2s = \sum_{i\in\mathcal{I}}\sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}}\left(v''_{in\bar{n}} - v'_{in}v''_{i\bar{n}y}\right) dx_{i\bar{n}}dx_{in} + p_{\underline{n}}\sum_{j\in\mathcal{J}}\sum_{n\in\mathcal{N}} \frac{df'_{jn}}{f'_{j\underline{n}}}dx_{jn}$$

$$d^2s = \sum_{i\in\mathcal{I}}\sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}}\left(v''_{in\bar{n}} - v'_{in}v''_{i\bar{n}y}\right) dx_{i\bar{n}}dx_{in} + p_{\underline{n}}\sum_{j\in\mathcal{J}} \frac{1}{f'_{j\underline{n}}}\sum_{n\in\mathcal{N}}\sum_{\bar{n}\in\mathcal{N}} f''_{jn\bar{n}}dx_{j\bar{n}}dx_{jn}$$

■

C.4 The loss

C.4.1 A first approximate expression of the loss

Proof. Let's consider the intermediate allocation $\bar{a}$ belonging to the path (C), as well as the isohedonic (surplus releasing) reallocation $\bar{\Delta}a = \bar{a} - a$. The releasable surplus attached to $\bar{\Delta}a$ can be written as a sum of infinitesimal releasings of surplus

$$\bar{\Delta}s(a) = \int_{\mathrm{a}=a}^{\bar{a}} ds(\mathrm{a}) \tag{303}$$

with

$$\bar{\Delta}s(a) = \Delta^* s(a) = l(a) \text{ for } \bar{a} = a^*. \tag{304}$$

Let's write the transfer of good n from $\hat{k}$ to k associated to $\bar{\Delta}a$ as a sum of infinitesimal transfers

$$\bar{\Delta}_{\hat{k}}x_{kn} = \int_{a}^{\bar{a}} d_{\hat{k}}x_{kn}. \tag{305}$$

Along the path (C) : $\left|v'_{kn}(\bar{a}) - v'_{\hat{k}n}(\bar{a})\right|$ decreases from $\left|v'_{kn}(a) - v'_{\hat{k}n}(a)\right| \geq 0$ to $\left|v'_{kn}(a^*) - v'_{\hat{k}n}(a^*)\right| = 0$; $\bar{\Delta}_{\hat{k}}x_{kn}$ varies from 0 in $\bar{a} = a$ to $\Delta^*_{\hat{k}}x_{kn}$ in $\bar{a} = a^*$. Along (C), differences $v'_{kn}(\bar{a}) - v'_{\hat{k}n}(\bar{a})$ are, at the second order approximation, linear functions of quantities $\bar{\Delta}_{\hat{k}}x_{kn}$. Among all the possible ways to move on the path (C), let's consider the process defined, for all $\left(k, \hat{k}\right) \in \mathcal{K}$, $\hat{k} > k$ and $n \in \mathcal{N}$ by

$$\bar{\Delta}_{\hat{k}}x_{kn} = \alpha \cdot \Delta^*_{\hat{k}}x_{kn}, \tag{306}$$

$\alpha \in [0; 1] : \alpha = 0$ corresponds to $\bar{a} = a$ while $\alpha = 1$ realizes $\bar{a} = a^*$. One gets:

$$d_{\hat{k}}x_{kn} = \Delta^*_{\hat{k}}x_{kn}d\alpha. \tag{307}$$

At the second order approximation, for all $\left(k, \hat{k}\right) \in \mathcal{K}$, $\hat{k} > k$ and $n \in \mathcal{N}$:

$$v'_{kn}(\bar{a}) - v'_{\hat{k}n}(\bar{a}) \sim \left(v'_{kn}(a) - v'_{\hat{k}n}(a)\right)(1 - \alpha). \tag{308}$$

At the third order approximation thus:

$$\bar{\Delta}s\left(a\right)\sim\int\limits_{\mathrm{a}=a}^{\bar{a}}\sum_{n\in\mathcal{N}}\sum_{k\in\mathcal{K}}\sum_{\hat{k}>k}\underbrace{\left(v_{kn}'\left(a\right)-v_{\hat{k}n}'\left(a\right)\right)\left(1-\alpha\left(\mathrm{a}\right)\right)}_{v_{kn}'\left(\bar{a}\right)-v_{\hat{k}n}'\left(\bar{a}\right)}\underbrace{\Delta_{\hat{k}}^{*}x_{kn}d\alpha\left(\mathrm{a}\right)}_{d_{\hat{k}}x_{kn}},\tag{309}$$

For $\bar{a}=a^{*}$:

$$\begin{aligned}\Delta^{*}s\left(a\right) &\sim \int\limits_{\alpha=0}^{1}\sum_{n\in\mathcal{N}}\sum_{k\in\mathcal{K}}\sum_{\hat{k}>k}\left(v_{kn}'\left(a\right)-v_{\hat{k}n}'\left(a\right)\right)\left(1-\alpha\right)\Delta_{\hat{k}}^{*}x_{kn}d\alpha,\\ \Delta^{*}s\left(a\right) &\sim \left(\int\limits_{\alpha=0}^{1}\left(1-\alpha\right)d\alpha\right)\sum_{n\in\mathcal{N}}\sum_{k\in\mathcal{K}}\sum_{\hat{k}>k}\left(v_{kn}'\left(a\right)-v_{\hat{k}n}'\left(a\right)\right)\Delta_{\hat{k}}^{*}x_{kn}.\end{aligned}$$

But $\int\limits_{0}^{1}\left(1-\alpha\right)d\alpha=\frac{1}{2}$ so that,

$$\Delta^{*}s\left(a\right)\sim\frac{1}{2}\sum_{n\in\mathcal{N}}\sum_{k\in\mathcal{K}}\sum_{\hat{k}>k}\left(v_{kn}'\left(a\right)-v_{\hat{k}n}'\left(a\right)\right)\Delta_{\hat{k}}^{*}x_{kn}.$$

■

C.4.2 Other expressions of the loss

Proof. Starting from the writing of the previous proposition:

$$l\left(a\right)\sim\frac{1}{2}\sum_{n\in\mathcal{N}}\sum_{k\in\mathcal{K}}\sum_{\hat{k}>k}\left(v_{kn}'\left(a\right)-v_{\hat{k}n}'\left(a\right)\right)\Delta_{\hat{k}}^{*}x_{kn},$$

one has

$$l\left(a\right)\sim\frac{1}{2}\sum_{n\in\mathcal{N}}\left(\begin{array}{c}\sum_{i\in\mathcal{I}}\sum_{\hat{\imath}>i}\left(v_{in}'\left(a\right)-v_{\hat{\imath}n}'\left(a\right)\right)\Delta_{\hat{\imath}}^{*}x_{in}\\ +\sum_{j\in\mathcal{J}}\sum_{\hat{\jmath}>j}\left(f_{jn}'\left(a\right)-f_{\hat{\jmath}n}'\left(a\right)\right)\Delta_{\hat{\jmath}}^{*}x_{jn}\\ +\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\left(v_{in}'\left(a\right)-f_{jn}'\left(a\right)\right)\Delta_{j}^{*}x_{in}\end{array}\right),$$

or

$$l\left(a\right)\sim\frac{1}{2}\sum_{n\in\mathcal{N}}\left(\begin{array}{c}\sum_{i\in\mathcal{I}}\sum_{\hat{\imath}\neq i}v_{in}'\left(a\right)\Delta_{\hat{\imath}}^{*}x_{in}\\ +\sum_{j\in\mathcal{J}}\sum_{\hat{\jmath}\neq j}f_{jn}'\left(a\right)\Delta_{\hat{\jmath}}^{*}x_{jn}\\ +\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}v_{in}'\left(a\right)\Delta_{j}^{*}x_{in}-\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}f_{jn}'\left(a\right)\Delta_{j}^{*}x_{in}\end{array}\right),$$

and thus

$$
\begin{aligned}
l(a) &\sim \frac{1}{2}\sum_{n\in\mathcal{N}}\left(\begin{array}{c} \sum_{i\in\mathcal{I}}\sum_{\hat{\imath}\neq i} v'_{in}(a)\,\Delta^*_{\hat{\imath}}x_{in} + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} v'_{in}(a)\,\Delta^*_{j}x_{in} \\ +\sum_{j\in\mathcal{J}}\sum_{\hat{\jmath}\neq j} f'_{jn}(a)\,\Delta^*_{\hat{\jmath}}x_{jn} + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} f'_{jn}(a)\,\Delta^*_{i}x_{jn} \end{array}\right), \\
l(a) &\sim \frac{1}{2}\sum_{n\in\mathcal{N}}\left(\begin{array}{c} \sum_{i\in\mathcal{I}}\sum_{k\neq i} v'_{in}(a)\,\Delta^*_{k}x_{in} \\ +\sum_{j\in\mathcal{J}}\sum_{k\neq j} f'_{jn}(a)\,\Delta^*_{k}x_{jn} \end{array}\right), \\
l(a) &\sim \frac{1}{2}\sum_{n\in\mathcal{N}}\left(\begin{array}{c} \sum_{i\in\mathcal{I}} v'_{in}(a)\sum_{k\neq i}\Delta^*_{k}x_{in} \\ +\sum_{j\in\mathcal{J}} f'_{jn}(a)\sum_{k\neq j}\Delta^*_{k}x_{jn} \end{array}\right), \\
l(a) &\sim \frac{1}{2}\sum_{n\in\mathcal{N}}\left(\sum_{i\in\mathcal{I}} v'_{in}(a)\,\Delta^* x_{in} + \sum_{j\in\mathcal{J}} f'_{jn}(a)\,\Delta^* x_{jn}\right).
\end{aligned}
$$

■